Commun Nonlinear Sci Numer Simulat 61 (2018) 84–103

Contents lists available at ScienceDirect

Commun Nonlinear Sci Numer Simulat

journal homepage: www.elsevier.com/locate/cnsns

Research paper

Transport structures in a 3D periodic flow

Henry Chang[a,*], Helga S. Huntley[a], A.D. Kirwan Jr.[a], B.L. Lipphardt Jr.[b], Mohamed H.M. Sulman[c]

[a] School of Marine Science and Policy, University of Delaware, Newark, DE, 19716, USA
[b] Newark, DE, 19711, USA
[c] Department of Mathematics and Statistics, Wright State University, Dayton, OH 45435, USA

ARTICLE INFO

Article history:
Received 10 August 2017
Revised 12 January 2018
Accepted 24 January 2018
Available online 31 January 2018

Keywords:
Lagrangian coherent structures
Transport boundaries
Finite-time Lyapunov exponents
Stratified 3D Euler flow
Internal wave
Super-inertial

ABSTRACT

The linearized 3D Euler equations on an *f*-plane with constant stratification admit a family of analytical wave solutions. Here, we investigate the Lagrangian properties of one such solution, a standing wave quadrupole, whose simplicity and symmetry make it an ideal time-varying 3D testbed for developing dynamical systems methods. In spite of its simplicity, the Eulerian solution gives rise to highly complex transport structures. Particle trajectories wind around tori-like surfaces with varying cross-sections. They are generally governed by the internal wave frequency plus subinertial frequencies, which depend on starting locations. The spatial variation in this subinertial period produces mixing in the periodic wave motion, a process completely distinct from diapycnal mixing typically associated with internal waves. Nonetheless, finite-time Lyapunov exponents, calculated from the 3D velocity field, clearly delineate transport barriers. These barriers identify five types of coherent Lagrangian structures, which oscillate at the super-inertial internal wave frequency. Two of these types are solely located near the surface, extending to depths unassociated with any Eulerian flow characteristic. The discovery of such shallow structures in the absence of a related Eulerian signal raises the interesting question whether similar structures may be hiding in the real ocean.

1. Introduction

Transport properties have been recognized as an important characteristic of fluid flows. They are inherently Lagrangian, and complex trajectory patterns often emerge even from simple Eulerian velocity fields. Thus, Eulerian velocities can be notoriously misleading indicators of transport mechanisms. This fact was already recognized by Cauchy and his contemporaries [1], but received little attention until numerical integration of the velocity field became more tractable [e.g., [2]]. Since then deriving Lagrangian properties from Eulerian velocities has become a topic of considerable interest.

Dynamical systems methods have proven to be particularly successful for this enterprise. A standard toy model for testing and illustrating these methods is a 2D multipole vortex system in a box. The vortex system is described by a Hamiltonian or streamfunction, whose flow symmetries typically are broken by a prescribed time-dependent perturbation.

* Corresponding author.
E-mail addresses: changh@udel.edu (H. Chang), helgah@udel.edu (H.S. Huntley), adk@udel.edu (A.D. Kirwan Jr.), bruce.lipphardt@gmail.com (B.L. Lipphardt Jr.), mohamed.sulman@wright.edu (M.H.M. Sulman).

https://doi.org/10.1016/j.cnsns.2018.01.014

Nearly all oceanographic applications of dynamical systems methods have been to large and mesoscale flows [e.g., [3–5]]. Here vanishingly small Rossby and Froude number dynamics rule. Consequently, these flows are essentially 2D and have a clear distinction between Eulerian and Lagrangian time-scales. It is hardly surprising then that early applications of dynamical systems methods to geophysical fluid dynamics relied on box models [3].

Of course, ocean mesoscale flows are baroclinic, and so stable and unstable manifolds in such flows are surfaces that may or may not be adequately represented by curves in the plane commonly used to depict them in early box models. There have been a few attempts to determine the vertical structure of manifolds in realistic flow settings, and they agree that these barriers tend to be slowly evolving and nearly vertical curtains [6–8]. This would imply that toy box models may still have some applicability as testbeds.

However, recent observations [9,10] suggest that submesoscale processes may erode the clear time and space divisions typical of mesoscale flows. On these scales, flows are 3D, their space- and time-scales range from 0.01 to 10 km and from hours to a few days, and the distinction between Eulerian and Lagrangian time-scales is blurred. This raises several fundamental questions. Two of concern here are: Can dynamical systems methods identify transport barriers in such flows, and how can one study the interaction of submesoscale and mesoscale flows?

A useful tool to address the first question would be an extension of the 2D box model to include both vertical and time dimensions. Velocity fields that could be used for this purpose are either generated by purely kinematic considerations or from solutions to the equations of motion. An early example of a kinematic model, designed specifically as a dynamical systems testbed, was proposed by Mezić and Wiggins [11]. More recent examples include those by Branicki and Wiggins [12] and Sulman et al. [13]. A limitation of kinematic models is that they do not account for dynamical processes. On the other hand, most dynamically based research focuses on applications to general circulation models (GCMs) [e.g., [7,8,14]] or approximations that tend to be restricted to mesoscale and comparable quasi-steady phenomena [e.g., [15,16]].

There are, of course, a large number of analytic solutions for Euler flows that might be useful for addressing submesoscale dynamics. Examples include the nonlinear ABC flow [17–20]; the Hill's vortex [21]; analytic multipolar solutions of the two-dimensional (2D) Euler equations [22–24]; new classes of exact solutions developed by Pukhnachov [25] and Aristov and Polyanin [26]; and Lagrangian solutions for time dependent flow fields [27]. These studies, however, are either restricted to 2D flow or to non-rotating frames, and none, to our knowledge, has continuous stratification. This restricts their use for benchmarking geophysical submesoscale flows. Staniforth and White [28,29] identified a similar need to test shallow water models in spherical and plane geometry for the atmosphere. Because of parameter range differences their solutions to the Euler equations have limited applicability to the ocean submesoscale.

To fill the need for a dynamically based testbed suitable for assessing submesoscale transport processes in an oceanic setting, we propose a class of solutions to the time-dependent, stratified, 3D, incompressible, linear Euler equations on the *f*-plane. A notable characteristic of the Euler equations in this setting is that they permit super-inertial fluctuations. Although internal waves are widely observed, and their time- and space-scales fall within the submesoscale regime, they are not currently considered an important submesoscale transport mechanism [30].

For present purposes the relative importance of internal waves in submesoscale transport is not of primary concern. Instead, our interest is in the fully 3D and transitory nature of their velocity field, as these are typical attributes of virtually all submesoscale processes. The primary questions of concern here are whether persistent transport barriers in a 3D time-dependent stratified flow can be detected by traditional dynamical systems methodologies, and if so, what their time- and space-scale characteristics are.

The model used to address these questions is summarized in Section 2, with full details provided in the appendix. The Eulerian properties of the velocity field are described in Section 3. Section 4 describes the Lagrangian properties and the transport barriers arising in this flow field. The last section summarizes and discusses some broader implications of the findings.

2. Model and solution

2.1. Euler equations on the *f*-plane

The linearized Boussinesq Euler equations for a stratified incompressible fluid on an *f*-plane are [cf. 31]

$$\frac{\partial u}{\partial t} - fv + \rho_0^{-1}\frac{\partial p}{\partial x} = 0, \tag{1a}$$

$$fu + \frac{\partial v}{\partial t} + \rho_0^{-1}\frac{\partial p}{\partial y} = 0, \tag{1b}$$

$$\frac{\partial w}{\partial t} + \rho_0^{-1}\frac{\partial p}{\partial z} + \frac{g\rho}{\rho_0} = 0, \tag{1c}$$

$$\frac{\partial \rho}{\partial t} + w\frac{\partial \rho_0}{\partial z} = 0, \tag{1d}$$