Commun Nonlinear Sci Numer Simulat 61 (2018) 138–148

Contents lists available at ScienceDirect

Commun Nonlinear Sci Numer Simulat

journal homepage: www.elsevier.com/locate/cnsns

Research paper

# A closed form expression for the Gaussian–based Caputo–Fabrizio fractional derivative for signal processing applications

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ARTICLE INFO

*Article history:*
Received 1 June 2017
Revised 19 December 2017
Accepted 27 January 2018

*Keywords:*
Fractional calculus
Gaussian-based derivatives
Fractional-order signal processing
Caputo–Fabrizio analytic solution

ABSTRACT

The Gaussian function has been employed in a vast number of practical and theoretical applications since it was proposed. Likewise, Gaussian function and its ordinary derivatives are considered as powerful tools for signal processing and control applications, *e.g.*, smoothing, sampling, change detection, blob detection, and transforms based on the Hermite polynomials. Nonetheless, it has impressive characteristics hidden amongst its fractional derivatives eager to be explored and studied in-depth. This work proposes a closed formula for the $(n+\nu)$–order fractional derivative of the Gaussian function, based on the Caputo–Fabrizio definition, as an approach for analysing those attributes. The obtained expression was numerically tested with several fractional orders, and their resulting behaviours were eventually analysed. Finally, three practical applications on signal processing via this closed formula were discussed, *i.e.*, customisable wavelets, image processing filters, and Rayleigh distributions.

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## 1. Introduction

Fractional calculus (FC) can be considered a mesmerising topic for the practical engineering community. It is defined as a generalisation of the traditional calculus, *i.e.*, derivatives and integrals are operated beyond integer-orders. Fractional calculus has evolved under the shadow of the traditional method, even if both emerged simultaneously, due to the apparent difficulty to compute fractional operators used in practical applications without the help of proper modern numerical methods. Furthermore, in some applications is hard to find a physical meaning to the fractional derivatives. However, scientists have been interested in FC applied for modelling natural phenomena since last century, like Leibniz prophesied, due to the increasing numerical power of modern computers [1].

FC has shown great capabilities for several classical applications, such as the tautochrone problem [2], the models based on memory mechanism [3], the space–time fractional diffusion equation [4], the new linear capacitor theory [5], the non–

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https://doi.org/10.1016/j.cnsns.2018.01.020
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local concepts in quantum mechanics (observed from the numerical solutions of the fractional Schrödinger equation) [6], to name a few. More examples of fractional calculus implementations are briefly described as follows. Brociek et al. solved numerically an inverse problem of the fractional order heat conduction with Robin boundary conditions [7]. Likewise, Sierociuk et al. studied the heat transfer process in a heterogeneous media using fractional partial differential equations [8]. Gómez–Aguilar et al. proposed a fractional model for electrical circuits [9]. Besides, Rosales–García et al., investigated some traditional phenomena of classical mechanics in the fractional sense, like the particle motion in different media, *e.g.,* empty–space and air [10,11]. Other FC applications have aimed at signal processing methodologies, such as Kawala–Janik et al. implemented a set of bi–fractional filters for electroencephalography signal analyses [12]. Kumar and Rawat presented a designing procedure for the Finite Impulse Response of fractional–order filters [13]. Specifically, FC has been used for enhancing and restoring image quality, for example, Jalab et al., proposed fractional–differential mask for texture enhancement, showing that fractional operators can extract subtle information and reinforce edge detection [14]. Pan et al., presented a competitive fractional integral low–pass filter tested on Computed Tomography, Magnetic Resonance, and Ultrasound medical images; this operator can preserve detailed edges and improves texture features [15]. Moreover, FC has been recently applied in complex image denoising problems, improving the removal of Gaussian and Speckle noise [16].

On the other hand, there exist a considerable number of fractional derivatives and antiderivatives, most of them proposed as kernels of a common operator, such as Grünwald–Letnikov, Riemann–Liouville [6], Caputo [17] and conformable derivatives [18,19]. Although, Caputo based derivatives have become extensively popular in different applications, specifically, the Caputo–Fabrizio (CF) derivative [20–22]. It was proposed as a derivative with a smooth and non-singular kernel [17,22]. Additionally, the well-known Gaussian distribution (or function), and its derivatives are broadly employed in multiple disciplines around the world, specifically in signal processing. Some few examples are the multi-scale filtering of vascular images [23], the edge detection of images [24], the semantic classification of remote sensing images [25], the machinery fault diagnosis through time-frequency analysis methods [26], and the many-body localisation edge in quantum systems [27].

This study presents a closed expression for the $(n+\nu)$–order fractional derivative of the Gaussian function, based on the CF definition. Several simulations highlight interesting features of the deduced formula, which can be implemented in a wide range of applications using Gaussian–based filtering. Hence, three signal processing applications for the obtained fractional derivative approach are studied and discussed. First application concerns to feature detection from digital signals by using wavelets based on the proposed expression. For illustrative purposes, an alternative strategy to identify an arrhythmia in ECG signals is discussed. Second application concerns image processing applications including colour and contrast improvement. Moreover, for segmentation contour extraction is improved using the non-symmetrical fractional derivatives and artefacts are detected on medical imaging. Last, but not least, introduces a probability density function based on the Rayleigh function and the Caputo–Fabrizio fractional derivative, for probabilistic filtering applications.

The manuscript is organised as follows: Section 1 introduces the Caputo fractional derivative family, focusing on the CF definition. Section 2 presents a brief mathematical deduction of the $(n+\nu)$–order fractional derivative of the Gaussian function and the analysis of several simulations. Subsequently, Section 3 discusses three practical applications for the obtained expression on signal processing, *i.e.,* customisable wavelets, image processing filters, and Rayleigh distributions. Finally, Conclusions in Section 5 display the main implications of this work.

## 2. Theoretical framework

This section introduces the generalised Caputo fractional derivative [28], as well as the specific Caputo–Fabrizio derivative [17]. This operator has been successfully used in many fields because it employs integer–order initial and boundary conditions, in fractional–order differential equation problems [1]. Hence, the $\nu$–order fractional derivative in the Caputo sense of a real, absolutely continuous, and causal function, $f(t):\mathbb{R}\to\mathbb{R}$, which is expressed as,

$$ {}^{C}D_t^{\nu} f(t) = \int_0^t k_\nu(t-\tau)\, \partial_t f(t)|_{t=\tau} d\tau, \tag{1}$$

where $\nu \in [0, 1]$, $\tau \in [0, t]$, $\partial_t\{\cdot\}$ is the ordinary first derivative operator, and $k_\nu(t)$ is the kernel following the properties given in [28]. Multiple kernels have been proposed in literature but, for the sake of brevity, only three are presented as follows.

The first fractional kernel, $k_\nu^{CD}(t)$, independently proposed by Caputo and Dzhrbashyan [1,28], is given by

$$ k_\nu^{CD}(t) = \frac{t^{-\nu}}{\Gamma(1-\nu)}, \tag{2}$$

since $\Gamma(\cdot)$ is the well–known Gamma function. However, Caputo and Fabrizio showed that this kernel has a singularity at $t=0$ [17]. Therefore, they proposed a non–singular kernel which follows Caputo derivative conditions [28]. This Caputo–Fabrizio (CF) fractional kernel, $k_\nu^{CF}(t)$, is described by the following formula,

$$ k_\nu^{CF}(t) = \frac{M(\nu)}{1-\nu} \exp\left(-\frac{\nu}{1-\nu} t\right), \tag{3}$$

with $M(\nu)$ as a normalisation function. It is assumed by several authors as $M(\nu)=1$. Losada and Nieto reported an interesting proof why that value is preferred [22]. Later, Atangana and Baleanu in [29] studied the Caputo–Fabrizio kernel locality. They proposed an alternative kernel with non–local and non–singular characteristics. Thus, Atangana–Baleanu–Caputo (ABC)

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