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Research paper

On the continuation of degenerate periodic orbits via normal form: full dimensional resonant tori

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ABSTRACT

We reconsider the classical problem of the continuation of degenerate periodic orbits in Hamiltonian systems. In particular we focus on periodic orbits that arise from the breaking of a completely resonant maximal torus. We here propose a suitable normal form construction that allows to identify and approximate the periodic orbits which survive to the breaking of the resonant torus. Our algorithm allows to treat the continuation of approximate orbits which are at leading order degenerate, hence not covered by classical averaging methods. We discuss possible future extensions and applications to localized periodic orbits in chains of weakly coupled oscillators.

1. Introduction

We consider a canonical system of differential equations with Hamiltonian

$$H(I, \varphi, \varepsilon) = H_0(I) + \varepsilon H_1(I, \varphi) + \varepsilon^2 H_2(I, \varphi) + \dots, \tag{1}$$

where $I \in \mathcal{U} \subset \mathbb{R}^n$, $\varphi \in \mathbb{T}^n$ are action-angle variables and ε is a small perturbative parameter. The unperturbed system, H_0, is clearly integrable and the orbits, lying on invariant tori, are generically quasi-periodic. Besides, if the unperturbed frequencies satisfy resonance relations, one has periodic orbits on a dense set of resonant tori.

The KAM theorem ensures the persistence of a set of large measure of quasi-periodic orbits, lying on nonresonant tori, for the perturbed system, if ε is small enough and a suitable nondegeneracy condition for H_0 is satisfied.

Instead, considering a resonant torus, when a perturbation is added such a torus is generically destroyed and only a finite number of periodic orbits are expected to survive. The location and stability of the continued periodic orbits are determined by a theorem of Poincaré [35,36], who approached the problem locally: with an averaging method, he was able to select those isolated unperturbed solutions which, under a suitable nondegeneracy condition (nowadays called Poincaré nondegeneracy condition), can be continued by means of an implicit function theorem. A modern approach has been developed in the seventies by Weinstein [41] and Moser [28] using bifurcation techniques, turning the problem to the investigation of critical points of a functional on a compact manifold. Actually, the number of critical points can be estimated from below with geometrical methods, like Morse theory. The drawback lies in the fact that the method is not at all constructive, thus it does not permit the localization of the periodic orbits on the torus. In the same spirit, variational methods which make use of the mountain pass theorem were developed some years later by Fadell and Rabinowitz, under different hypotheses (see Chapter 1 in [4] for a simplified exposition of this result). More recently, the problem of continuation of degenerate

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periodic orbits in nearly integrable Hamiltonian systems using perturbation techniques has been studied in [25,40]. On the other hand, from the early nineties great attention has been devoted to the generalization of Poincaré's result to partially resonant tori, where the unperturbed torus is foliated by quasi-periodic orbits, since the number of resonances is strictly less than $n-1$. In this case, the starting point still consists in looking for nondegenerate critical points of the perturbation averaged over the unperturbed quasi-periodic solution; however, the presence of more than a single frequency requires the assumption of additional hypotheses, which allow to implement suitable versions of the KAM scheme. Along this line, first results were due to Treshchev [39], Cheng [6], Li and Yi [24]. Recently, these results have been successfully extended to multiscale nearly integrable Hamiltonian systems, where the integrable part of the Hamiltonian $H_0(I, \varepsilon)$, properly involves several time scales, see, e.g., [42,43]. All the quoted works deal with the case where the unperturbed invariant torus is degenerate due to resonances among its frequencies. Instead, we remark that the problems of existence of invariant tori of dimension less than the number of degrees of freedom in weakly perturbed Hamiltonian system, i.e., the extension to lower dimensional tori of the classical KAM theory, has been widely investigated by many authors, see, e.g., [9,23,26,27,29,34,45–47] in a general abstract framework, and [5,7,8,14,17,38] for more recent problems mainly emerging in Celestial Mechanics.

In this paper we follow the line traced by Poincaré and deal with those cases when the nondegeneracy condition is not fulfilled. In particular, under a twist-like condition of the form (4) (see, e.g., [3]) and analytic estimates of the perturbation (5), we develop an original normal form scheme, inspired by a recent completely constructive proof of the classical Lyapunov theorem on periodic orbits [13], which allows to investigate the continuation of degenerate periodic orbits. Precisely, first we identify possible candidates for the continuation via normal form, then we prove the existence of a unique solution by using the Newton-Kantorovich method.

Remark 1.1. Let us anticipate a crucial difference with respect to the KAM normal form algorithm: generically, our normal form procedure turns out to be divergent. Actually, a moment's thought suggests that looking for a convergent normal form which is valid for all possible periodic orbits is too much to ask. The idea is that a suitably truncated normal form allows to produce the approximated periodic orbits and the continuation can be performed via contraction or with a further convergent normal form around a selected periodic orbit.

Remark 1.2. It is worth mentioning that the idea of performing a finite number of KAM-like steps in order to remove some degeneracy in the continuation procedure is obviously not new, see, e.g., [17,42,43] concerning the continuation of quasi-periodic orbits on resonant tori for a class of multiscale nearly integrable Hamiltonian systems. In these works a finite number of preliminary KAM steps are performed in order to push the perturbation to a sufficiently high order in ε, before applying a standard convergent scheme.

The strength of the present perturbative algorithm is at least twofold. First, it provides a way to construct approximate periodic solutions at any desired order in ε, thus going beyond the average approximation mostly used in the literature. One of the few results which represents an improvement with respect to the usual average method is the one claimed in [25], where a criterion for the existence of periodic orbits on completely degenerate resonant tori is proved. In that work the authors, by means of a standard Lindstedt expansion as the original works of Poincaré, are able to push the perturbation scheme at second order in the small parameter ε. However, the possibility to provide a criterion for the continuation, although remarkable, is a consequence of the restriction to completely degenerate cases, like when the Fourier expansion of H_1 with respect to the angle variables does not include a certain resonance class. In this way, all the partial degeneracies are excluded. Such a limitation is overcome by the normal form that we propose: indeed, by being able to deal with any degree of degeneracy, it results more general (also in terms of order of accuracy), thus including also the above mentioned result.

The formal scheme itself has also a second relevant aspect. Since this approximation is given by a recursive explicit algorithm, it can be much useful for numerical applications (see, e.g., [11]) and it is independent of the possibility to conclude the proof with a contraction theorem. Furthermore, our approach provides a constructive normal form that can be applied to a sufficiently general class of models; for example, it includes nonlinear Hamiltonian lattices with next-to-nearest neighbor interactions, such as

$$H = \sum_{j\in\mathcal{J}} \frac{y_j^2}{2} + \sum_{j\in\mathcal{J}} V(x_j) + \varepsilon \sum_{l=1}^{r} \sum_{j\in\mathcal{J}} W(x_{j+l} - x_j);$$

where $V(x)$ is the potential of an anharmonic oscillator which allows for action variable (at least locally, like the Morse potential), and $W(x)$ represents a generic next-to-nearest neighbour (possibly linear) interaction, with r the maximal range of the interaction. In this class of nearly integrable Hamiltonian lattices, the possibility to generalize the formal scheme to lower dimensional tori would represent a remarkable breakthrough in the investigation of degenerate phase-shift multibreathers and vortexes in one and two dimensional lattices (see, e.g.,[1,2,21,22,30–33]). The extension to lower dimensional tori, that represents the natural continuation of the present work, will be also useful in problems emerging in Celestial Mechanics, where the persistence of nonresonant lower dimensional tori has been proved with similar techniques, see, e.g., [14,38].

In the present work we focus on resonant maximal tori in order to reduce the technical difficulty to a minimum and concentrate on the novelty of the normal form scheme.