# Numerical continuation of invariant solutions of the complex Ginzburg-Landau equation 

Vanessa López<br>IBM Research, T. J. Watson Research Center, 1101 Kitchawan Road, Route 134, Yorktown Heights, NY 10598, USA

## A R T I C L E I N F O

## Article history:

Received 14 March 2017
Revised 19 December 2017
Accepted 27 January 2018
Available online 22 February 2018

## Keywords:

Invariant solutions
Complex Ginzburg-Landau equation
Continuous symmetries
Numerical continuation


#### Abstract

We consider the problem of computation and deformation of group orbits of solutions of the complex Ginzburg-Landau equation (CGLE) with cubic nonlinearity in $1+1$ space-time dimension invariant under the action of the three-dimensional Lie group of symmetries $A(x, t) \rightarrow \mathrm{e}^{\mathrm{i} \theta} A(x+\sigma, t+\tau)$. From an initial set of group orbits of invariant solutions, for a particular point in the parameter space of the CGLE, we obtain new sets of group orbits of invariant solutions via numerical continuation along paths in the moduli space. The computed solutions along the continuation paths are unstable, and have multiple modes and frequencies active in their spatial and temporal spectra, respectively. Structural changes in the moduli space resulting in symmetry gaining / breaking associated often with the spatial reflection symmetry $A(x, t) \rightarrow A(-x, t)$ of the CGLE were frequently uncovered in the parameter regions traversed.


© 2018 Elsevier B.V. All rights reserved.

## 1. Introduction

We consider the problem of numerical computation and deformation of solutions of evolutionary partial differential equations (PDEs) fixed by the action of a subgroup of a Lie group $\Gamma=\Gamma_{1} \times \mathbb{R}$ of continuous symmetries of the PDEs, where $\mathbb{R}$ is the group of time translations and $\Gamma_{1}$ is non-trivial. Within this context, such invariant solutions are also known as relative periodic orbits or relative time-periodic solutions of an (autonomous) equivariant dynamical system. In this paper, we work with the complex Ginzburg-Landau equation with cubic nonlinearity in $1+1$ space-time dimension, with $\Gamma_{1}=\mathbb{T}^{2}\left(=S^{1} \times S^{1}\right)-$ the two-torus. We note, however, that it should be straightforward to apply the methodology described in this paper to other evolutionary parameter-dependent PDEs invariant under the action of a group of continuous transformations.

The complex Ginzburg-Landau equation (CGLE) is a widely studied PDE which has become a model problem for the study of nonlinear evolution equations exhibiting chaotic spatio-temporal dynamics, as well as being of interest in the context of pattern formation. It has applications in various fields, including fluid dynamics and superconductivity. (For details see, for example, $[2,21,25,34]$ and references therein.) Following [23], we consider here the CGLE with cubic nonlinearity in one spatial dimension,

$$
\begin{equation*}
\frac{\partial A}{\partial t}=R A+(1+\mathrm{i} v) \frac{\partial^{2} A}{\partial x^{2}}-(1+\mathrm{i} \mu) A|A|^{2} \tag{1}
\end{equation*}
$$

with periodic boundary conditions

$$
\begin{equation*}
A(x, t)=A\left(x+L_{x}, t\right) \tag{2}
\end{equation*}
$$

[^0]

Fig. 1. Fibered space $(\mathfrak{M}, \mathcal{B}, \pi)$.
and spatial period $L_{x}=2 \pi$. The CGLE also appears in the literature in the form

$$
\begin{equation*}
\frac{\partial A}{\partial t}=A+(1+i v) \frac{\partial^{2} A}{\partial x^{2}}-(1+i \mu) A|A|^{2}, \quad A(x, t)=A(x+L, t) \tag{3}
\end{equation*}
$$

but note that with a change of variables $x \rightarrow\left(L_{x} / L\right) x, t \rightarrow\left(L_{x} / L\right)^{2} t, A \rightarrow\left(L_{x} / L\right) A$ one obtains Eq. (1), with $R=\left(L / L_{x}\right)^{2}$. Thus we adopt the formulation (1) without loss of generality and, henceforth, when we refer to the CGLE we mean Eq. (1) with the boundary conditions (2) unless otherwise noted.

Eq. (1) describes the time evolution of a complex-valued field $A(x, t)$. The parameters $R$, $v$, and $\mu$ in the equation are real. When $R>0$ there is, in general, nontrivial spatio-temporal behavior and this is therefore the region of interest. The parameters $v$ and $\mu$ are measures of the linear and nonlinear dispersion, respectively [2,21].

As will be discussed in detail in Section 2, the CGLE has a three-parameter group $G=\mathbb{T}^{2} \times \mathbb{R}$ of continuous symmetries generated by space-time translations and a rotation of the complex field $A$. Thus, we focus our study on invariant solutions of the CGLE, namely, the ones that in addition to (1) and (2) satisfy

$$
\begin{equation*}
A(x, t)=\mathrm{e}^{\mathrm{i} \varphi} A(x+S, t+T) \tag{4}
\end{equation*}
$$

for some $(\varphi, S) \in \mathbb{T}^{2}$ and $T>0$. The interest here is on invariant solutions of the CGLE having multiple frequencies active in their temporal spectrum, not on single-frequency solutions $A(x, t)=B(x) \mathrm{e}^{\mathrm{i} \omega t}[10,15,17]$ or generalized traveling waves $A(x, t)=\rho(x-v t) \mathrm{e}^{\mathrm{i} \phi(x-v t)} \mathrm{e}^{\mathrm{i} \omega t}$, where $\rho$ and $\phi$ are real-valued functions and $\omega$ is some frequency $[2,7,25,33]$, which have been considered more extensively than the multiple-frequency class. The CGLE is also invariant under the action of the discrete group of transformations $A(x, t) \rightarrow A(-x, t)$ and thus solutions of the CGLE may also be fixed by this $\mathbb{Z}_{2}$-symmetry. While it is not uncommon in studies to center on solutions fixed by the $\mathbb{Z}_{2}$-symmetry (for example, even solutions), we make no such restriction here in order to work with a broader solution space.

Since we are actually working with a 3-parameter family (1) of equations, this family defines implicitly a fibered space $\mathcal{S} \xrightarrow{p} \mathcal{B}$ over the space of parameters $\mathcal{B}=\{(R, \nu, \mu)\} \subset \mathbb{R}^{3}$, where $\mathcal{S}$ is the total space of solutions of (1) and $\mathcal{B}$ forms the base of the fibered space. Moreover, the group $G$ acts on the total space of solutions $\mathcal{S}$. Therefore, we consider the quotient fibered space $\mathfrak{M} \xrightarrow{\pi} \mathcal{B}$ modulo this action. Here $\mathfrak{M}=\mathcal{S} / \sim$ is the total moduli space, where $\sim$ is a relation between the points of $\mathcal{S}$ established by the group action which is compatible with $p$, that is, for any $s^{\prime}, s \in \mathcal{S}, s^{\prime} \sim s$ if and only if $p\left(s^{\prime}\right)=p(s)$ and there exists a $g \in G$ such that $s^{\prime}=g$.s. Then $\pi$ is the map induced by $p$ after taking the quotient, and the points of $\mathfrak{M}$ are in one-to-one correspondence with $G$-orbits whose elements are all mapped by $p$ to the same point in the base $\mathcal{B}$. Thus, geometrically we have a fibered space, that is, a triple ( $\mathfrak{M}, \mathcal{B}, \pi$ ), depicted in Fig. 1, whose fibers $\mathcal{M}_{R, v, \mu}=\pi^{-1}(R, v, \mu)$ over each point $(R, v, \mu) \in \mathcal{B}$ of the base are moduli spaces of solutions of the CGLE. Note that we do not know the explicit form of the map $\pi$. It is defined implicitly by Eq. (1). In essence, it is our goal to understand and reveal its properties. Therefore, the aim of the present study is to acquire a more global view of (a part of) the fibered space of G-orbits of the CGLE and its structure as we move around the point $(R, v, \mu)$ in the base space $\mathcal{B}$.

More precisely, and referring again to Fig. 1, here we are interested in the (sub)fibered space $(\mathcal{I}, \mathcal{B}, \pi \mid \mathcal{I}) \subset(\mathfrak{M}, \mathcal{B}, \pi)$, where the points of the subspace $\mathcal{I} \subset \mathfrak{M}$ are $G$-orbits of invariant solutions of the CGLE. Namely, these are solutions that satisfy, in addition to (1)-(2), the functional Eq. (4). Then the fiber of $\pi \mid \mathcal{I}, \mathcal{I}_{R, v, \mu}=(\pi \mid \mathcal{I})^{-1}(R, v, \mu) \subset \mathcal{M}_{R, v, \mu}$, over each point $(R, v, \mu) \in \mathcal{B}$ of the base is a moduli space of such invariant solutions of the CGLE. Note that a $G$-orbit in $\mathcal{I}_{R, v, \mu}$ is determined uniquely by a quadruple ( $A(x, t, R, v, \mu), \varphi(R, v, \mu), S(R, v, \mu), T(R, v, \mu)$ ), where $A(x, t, R, v, \mu)$ is an element of the orbit (that is, an invariant solution) over the point $(R, v, \mu) \in \mathcal{B}$. Further, for each point $(R, v, \mu) \in \mathcal{B}$ the space $\mathcal{I}_{R, v, \mu}$ is

# https://daneshyari.com/en/article/7154684 

Download Persian Version:

## https://daneshyari.com/article/7154684

## Daneshyari.com


[^0]:    E-mail address: lopezva@us.ibm.com

