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Research paper

Periodic and rational solutions of the reduced Maxwell–Bloch equations

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ABSTRACT

We investigate the reduced Maxwell–Bloch (RMB) equations which describe the propagation of short optical pulses in dielectric materials with resonant non-degenerate transitions. The general *N*th-order periodic solutions are provided by means of the Darboux transformation. The *N*th-order degenerate periodic and *N*th-order rational solutions containing several free parameters with compact determinant representations are derived from two different limiting cases of the obtained general periodic solutions, respectively. Explicit expressions of these solutions from first to second order are presented. Typical nonlinear wave patterns for the four components of the RMB equations such as single-peak, double-peak-double-dip, double-peak and single-dip structures in the second-order rational solutions are shown. This kind of the rational solutions correspond to rogue waves in the reduced Maxwell–Bloch equations.

1. Introduction

It is known that the Maxwell-Bloch (MB) equations can describe the propagation of ultra-short optical pulses in resonant media, and the associated reduced Maxwell-Bloch (RMB) equations are used to describe phenomena in nonlinear optics related to the theory of self-induced transparency (SIT) [1,2]. This SIT is firstly discovered by McCall and Hahn when they investigate the motion of an ultrashort light pulse in a resonant absorbing sample in experiments and observe the phenomenon of abnormally low energy loss with the optical pulse exceeding some critical value [3]. Whereafter, Lamb found that, the phase variation of coherent optical pulses in a two-level resonant medium to a reasonable approximation is usually described by the MB equations [4], which can be written as

$$E_{xx} - c^{-2}E_{tt} = 4\pi c^{-2} np\langle u_{tt}\rangle, \tag{1a}$$

$$u_t = -\omega_a v, \tag{1b}$$

$$v_t = \omega_a u + 2p\hbar^{-1}Ew, \tag{1c}$$

$$w_t = -2p\hbar^{-1}Ev, \tag{1d}$$

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where $E(x, t)$ represents the electric field, $u(x, t)$, $v(x, t)$ and $w(x, t)$ stand for the atomic dipole, phase information and atomic inversion, respectively, and the bracketed term $\langle \cdot \rangle$ means

$$\langle F \rangle = \int_0^{\infty} F(x, t; \omega_a) g(\omega_a) \mathrm{d}\omega_a,$$

with $g(\omega_a)$ being the spread of the inhomogeneous atomic resonance frequency. Here, n is the concentration of resonant atoms, p is the projection of a matrix element of the dipole operator on the direction of the constant polarization vector, c is the velocity of light in vacuum, ω_a is the resonance frequency, and $\hbar$ denotes the Planck constant.

Moreover, as is showed by Eilbeck [5], if the density of resonant atoms is small enough to make the parameter $4\pi np/\hbar\omega_a$ less than unity, then the interference between the oppositely directed waves may be neglected. In this circumstance, by assuming that each atom is to be at different resonance frequency ω_a', the MB equations are converted to the simpler RMB equations by approximation technique [6], namely,

$$E_x + c^{-1}E_t = -2\pi np\langle u_t \rangle, \tag{2a}$$

$$u_t = -\omega_a' v, \tag{2b}$$

$$v_t = \omega_a' u + 2p\hbar^{-1}Ew, \tag{2c}$$

$$w_t = -2p\hbar^{-1}Ev. \tag{2d}$$

It should be remarked that, in both the MB equations and RMB equations the symbol E denotes the real value of the electric field,

$$E(x, t) = \mathcal{E}(x, t)\cos[kx - \omega t - \phi(x, t)],$$

where $\mathcal{E}(x, t)$ is the pulse envelope, ω is the radiation frequency, k is the wave number, and $\phi(x, t)$ denotes the phase. When $g(\omega_a') = \delta(\omega_a' - \omega_a)$, one can see that $\langle u \rangle = \int_0^{\infty} u(x, t; \omega_a') g(\omega_a') \mathrm{d}\omega_a'$ is simplified to u. At this point, by scaling the RMB equations in the sharp-line limit through the transformation

$$\widehat{x} = t - c^{-1}x, \quad \widehat{t} = -4\pi np^2\hbar^{-1}\omega_a' x, \quad \widehat{E} = 2p\hbar^{-1}E,$$

and dropping the bars, we can obtain the scaled version of the sharp-line limit RMB equations with the form [2]

$$u_x = -\mu v, \tag{3a}$$

$$v_x = Ew + \mu u, \tag{3b}$$

$$w_x = -Ev, \tag{3c}$$

$$E_t = -v, \tag{3d}$$

Here we have set $\mu = \omega_a'$. The integrability such as the Lie-algebra-valued differential forms and Painlevé test of Eq. (3) have been investigated in Refs. [2,7], and the explicit N-soliton solutions of Eq. (3) and its relevant equations have been extensively studied by the inverse scattering transform, Hirota bilinear technique and Darboux transformation (DT) during the past few decades [8–16].

Recently, the generation of unexpectedly huge waves (termed as “rogue waves”) has received widespread attention in quite a lot of researches including oceanography, optical fields, Bose–Einstein condensates, plasma physics, etc. [17–22]. The straightforward description of a single rogue wave in mathematics is the Peregrine soliton [23], a special solution of the nonlinear Schrödinger (NLS) equation, which is a combination of the second-order rational polynomials and exponential function, and simulates the evolution of a wave of large amplitude that is localized in both space and time. More recently, beyond the NLS equation and its generalized physical systems [24–32], explicit periodic solutions, rational solutions and the generation of rogue waves in the modified Korteweg-de Vries (mKdV) equation have been studied by Chowdury, Slunyaev and He et al. [33–35]. As is pointed out by them, the existences of periodic and rational solutions in the mKdV equation reveal that breather and rogue wave phenomena are not confined to the deep ocean, and rogue wave phenomena governed by the mKdV equation present quite different descriptions in hydrodynamics from that related to the NLS equation.

In this paper, we demonstrate that Eq. (3) can also possess periodic and rational solutions like the mKdV equation, which will be helpful to understand the complicated rogue wave phenomena in nonlinear optics governed by the RMB equations. We present the general Nth-order periodic solutions on a finite constant background by using the classical DT with N eigenvalues that are different from each other [36–39]. Then, by taking advantage of the limit approach, namely the generalized DT [40–49], the Nth-order degenerate periodic and Nth-order rational solutions in the compact determinant representations can be respectively derived from two kinds of limiting cases of the general periodic solutions. As an application, explicit periodic, degenerate periodic and rational solutions up to second order are presented. We hereby show that the doubly-periodic lattice-like structure, and the single periodic- peaks or dips on a periodic wave background structure can exist