Commun Nonlinear Sci Numer Simulat 59 (2018) 15–29

Contents lists available at ScienceDirect

Commun Nonlinear Sci Numer Simulat

journal homepage: www.elsevier.com/locate/cnsns

Research paper

Recurrence quantity analysis based on matrix eigenvalues

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ARTICLE INFO

Article history:
Received 30 November 2016
Revised 20 October 2017
Accepted 1 November 2017

Keywords:
Recurrence plots
Recurrence quantification analysis
Entropy
Eigenvalues

ABSTRACT

Recurrence plots is a powerful tool for visualization and analysis of dynamical systems. Recurrence quantification analysis (RQA), based on point density and diagonal and vertical line structures in the recurrence plots, is considered to be alternative measures to quantify the complexity of dynamical systems. In this paper, we present a new measure based on recurrence matrix to quantify the dynamical properties of a given system. Matrix eigenvalues can reflect the basic characteristics of the complex systems, so we show the properties of the system by exploring the eigenvalues of the recurrence matrix. Considering that Shannon entropy has been defined as a complexity measure, we propose the definition of entropy of matrix eigenvalues (EOME) as a new RQA measure. We confirm that EOME can be used as a metric to quantify the behavior changes of the system. As a given dynamical system changes from a non-chaotic to a chaotic regime, the EOME will increase as well. The bigger EOME values imply higher complexity and lower predictability. We also study the effect of some factors on EOME,including data length, recurrence threshold, the embedding dimension, and additional noise. Finally, we demonstrate an application in physiology. The advantage of this measure lies in a high sensitivity and simple computation.

1. Introduction

For a large number of scientific disciplines, such as astrophysics, biology or geosciences, we can do some data analysis to understand the complex process observed in nature. Different types of systems, from very large to very small time scales can be modeled by differential equations. In principle, we can predict the state of such a system with arbitrary precision once the initial conditions are known. It is because these systems always evolve in a similar way or occur over and over again [1,2]. However, some complex systems are very sensitive to fluctuations and even the smallest perturbations of the initial conditions can make a precise prediction on long time scales impossible. Linear approaches of time series analysis are often insufficient, and most nonlinear techniques [3–8], such as fractal dimensions [9] or Lyapunov exponents [10,11], suffer from the curse of dimensionality and need long data series. Therefore, the application of these methods, especially for short time series, can lead to serious pitfalls. Entropy, to a certain extent, can show the complexity of a system, such as the Rényi entropy [12], the renormalized entropy [13] and so on.

In this paper, we focus on another technique of complexity measure, which is based on the method of recurrence plots (*RPs*). In 1987, the method of RPs was first introduced by Eckmann et al. to visualize the recurrences of dynamical systems, which can be portrayed by a trajectory $\{\vec{x}_i\}_{i=1}^{N}$ in its phase space [14]. Then, the corresponding RP is defined as the following

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https://doi.org/10.1016/j.cnsns.2017.11.001
1007-5704/

matrix:

$$R(i,j)=\begin{cases}1 & \vec{x_i}\approx\vec{x_j}\\ 0 & \vec{x_i}\neq\vec{x_j}\end{cases}\quad i,j=1,2,\ldots,N \tag{1}$$

where $\vec{x_i}\approx\vec{x_j}$ means equality up to a cut-off distance ε. That is to say, the matrix reflects whether the states of a system at times i and j are similar or not. If the states are similar, $R(i,j)=1$. On the contrary, if the states are rather different, the corresponding entry in the matrix $R(i, j)$ is 0. So this matrix can tells us when the system's state will appear again. This approach has been used to analyze non-stationary and short data series [15]. RPs have been considered as a powerful technique to reveal the statistical properties of the system through the structural features of RPs. Cross recurrence plot (CRP) [16,17] and Joint recurrence plot (JRP) [18,19] were proposed as bivariate extension of the RP. CRPs reveal valuable information about the relationship between both systems by comparing their states represented in two time series. JRPs compare different systems by considering the recurrences of their trajectories in their phase spaces separately and look for the times when both of them recur simultaneously.

Beyond the visual impression yielded by RPs, some measures for recurrence plots have been proposed and used to detect typical transitions occurring in complex systems, which are known as recurrence quantification analysis (RQA). RQA reflects the nonlinear properties over the recurrence point density, the diagonal and the vertical line of the RPs [20–26]. RQA contains several measures: measures based on the recurrence density—recurrence rate (RR), measures based on diagonal lines—determinism (DET), the average diagonal line length (L), divergence (DIV), entropy (ENTR) and so on; measures based on vertical lines—laminarity (LAM), trapping time (TT) and the maximal length of the vertical lines (Vmax). Compared to many classical nonlinear analysis methods, RQA overcomes some limitations, which does not require large data sizes and is less affected by noise and non-stationarity [20,27].

RQA based on structural characteristics of RP is very mature. Recurrence quantity analysis based on singular value decomposition is studied in [28]. Considering RP is a symmetric and binary matrix and matrix eigenvalues carry a lot of basic information about the complex systems, we study the RP from the aspect of matrix eigenvalue. There are many eigenvalues of RP close to zero, and the greater the proportion of eigenvalues approaching zero is, the higher the stability and predictability of the system are. For example, the ratio of the eigenvalues close to zero for periodic system is bigger than that of the non-periodic system. Because the eigenvalues close to zero are quite different in the order of magnitude, we put all the eigenvalues of the logarithmic scale. In general, the eigenvalues can be divided into three parts by a logarithmic scale. Our idea is to reflect the characteristics of the system through the complexity of eigenvalues of RP. The Shannon entropy is a measure to assess the complexity of a dynamical process and can be used to quantify transitions between different dynamical regimes, so we put forward the entropy of eigenvalues (EOME) for RPs based on Shannon entropy.

We verify that the measure EOME can quantitatively describe the RPs through experiments, and it is especially helpful to find various transitions in dynamical systems. For periodic system the value of EOME is rather small, indicating its low complexity and high predictability. However, with an increasing chaotic nature of the system the EOME values will increase. We find that EOME is sensitive to noise, so it is necessary to ensure that the data is not disturbed by noise. The influence of other factors on EOME is also studied. Logistic map, as a typical example of complex systems, is used by us to verify that EOME can measure the characteristics of different systems. Compared with other RQA metrics, EOME is a better choice to detect the transitions from periodic to chaotic and chaotic to periodic states.

This paper is organized as follows. In Section 2, the definitions of RP and EOME are proposed. In addition, we introduce some features of RPs and how to calculate EOME of RP. Section 3 is devoted to prove EOME can be used to distinguish different systems. The effects of some factors on EOME are also studied in this section. In Section 4, the logistic map is used to demonstrate that EOME can detect the transitions between different systems. Furthermore, we demonstrate an application of EOME in physiology. We summarize and give our conclusions in Section 5.

2. Methodology

2.1. Recurrence plots

In our daily life, some situations occur over and over again. Similarly, in some systems, some conditions occur over and over again. Recurrence plot is a tool which measures recurrences of a state. Recall that we have a time series $\{u_i\}_{i=1}^N$ with the length N. After choosing the time delay τ and embedding dimension m, we can express the dynamics with a reconstruction of the phase space trajectory $\vec{x_t}$ from a time series $\{u_i\}_{i=1}^N$ [29,30]:

$$\vec{x_t}=(u_t,u_{t+\tau},\ldots,u_{t+(m-1)\tau}),\qquad t=1,2,\ldots,N-(m-1)\tau \tag{2}$$

Two embedded parameters, the dimension m and delay τ, must be chosen appropriately. Methods for the estimation of the smallest sufficient embedding dimension (e.g. false nearest neighbours [31]) and for an appropriate time delay (e.g. the auto-correlation function, the mutual information function [32,33]) have been proposed.

In phase space, for a given trajectory $\vec{x_i}$, the recurrence plots are defined as [14,34]:

$$R_{i,j}(\varepsilon)=\Theta(\varepsilon-||\vec{x_i}-\vec{x_j}||),\qquad i,j=1,2,\ldots,N \tag{3}$$

where $\vec{x_i}$ is a state point of a system in a phase space, ε is a threshold distance, $\Theta(\cdot)$ is the Heaviside function and $||\cdot||$ is a norm. The most frequently used norms are the L_1-norm, the L_2-norm (Euclidean norm) and the L_∞-norm (Maximum or