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Backward Reachability of Autonomous Max-Plus-Linear Systems ⋆

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Abstract: This work discusses the backward reachability of autonomous Max-Plus-Linear (MPL) systems, a class of continuous-space discrete-event models that are relevant for applications dealing with synchronization and scheduling. Given an MPL system and a continuous set of final states, we characterize and compute its "backward reach tube" and "backward reach sets," namely the set of states that can reach the final set within a given event interval or at a fixed event step, respectively. We show that, in both cases, the computation can be done exactly via manipulations of difference-bound matrices. Furthermore, we illustrate the application of the backward reachability computations over safety and transient analysis of MPL systems.

Keywords: Backward reachability analysis, backward reach tube, max-plus-linear systems, piecewise affine systems, difference-bound matrices, safety and transient analysis

1. INTRODUCTION

Max-Plus-Linear (MPL) systems are discrete-event models (Baccelli et al., 1992; Hillion and Proth, 1989; Cuninghame-Green, 1979) with a continuous state space characterizing the timing of the underlying discrete events. MPL systems are predisposed to describe the timing synchronization between interleaved processes, under the assumption that timing events are linearly dependent (within the max-plus algebra) on previous event occurrences (cf. Section 2). These models are widely employed in the analysis and scheduling of infrastructure networks, such as communication and railway systems (Heidergott et al., 2006), production and manufacturing lines (Roset et al., 2005; van Eekelen et al., 2006), as well as in biological systems (Brackley et al., 2012). They cannot model concurrency and are related to a subclass of timed Petri nets, namely timed-event graphs (Baccelli et al., 1992).

Reachability analysis of MPL systems from a *single* initial condition has been investigated in (Gazarik and Kamen, 1999; Gaubert and Katz, 2003), by computing the reachability matrix as in the case of discrete-time linear dynamical systems. It has been shown in (Gaubert and Katz, 2006, Sec. 4.13) that the reachability problem for autonomous MPL systems with a single initial condition is decidable – this result however does not hold for a general, uncountable set of initial conditions. Furthermore, the existing literature does not deal with backward reachability analysis, which would require expressing the set of final conditions as a max-plus convex cone (Gaubert and Katz, 2007). Furthermore, the computation would need the system matrix to be max-plus invertible. A matrix is max-plus invertible iff there is a single finite element (not equal to $-\infty$) in each row and in each column. In conclusion, these assumptions limit the applicability of the approach.

In this work, we extend the state-of-the-art results in backward reachability analysis of MPL systems by presenting a computational approach that can handle state matrices that are not max-plus invertible and further manage problems over an arbitrary (possibly uncountable) set of final conditions. We start by characterizing MPL systems alternatively by Piece-wise Affine (PWA) systems, and show that the dynamics can be fully represented by Difference-Bound Matrices (DBM) (Dill, 1990, Sec. 4.1), which are structures that are quite simple to manipulate computationally. Furthermore, one can show that DBM are closed under PWA dynamics, which leads to being able to compute a set of states that is mapped to given DBM-sets through an MPL system. Given a set of final states, we then compute its "backward reach tube" and the collection of "backward reach sets," namely the set of states that can arrive at the final states in any number of steps and in a fixed number of steps, respectively. We further describe two alternative approaches to compute the latter quantities.

Closely related to backward reachability is the problem of safety analysis (Mitchell, 2007): given an unsafe set over the state space, it is of interest to determine whether trajectories of the model enter the unsafe set – either at a given event step, or over an events interval. If the model

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is not safe, we can seek the subset of initial conditions leading to the unsafe set by using backward reachability analysis.

In addition to general safety analysis, we show that backward reachability is specifically helpful in the transient analysis of MPL systems. According to the max-plus algebra analogue of the Perron-Frobenius theorem (Baccelli et al., 1992, Sec. 3.7), if the system matrix is irreducible, there exists a periodic behavior ensuing after some event index. The smallest of such indices is called the length of the transient part, which is used in the literature to characterize model performance. For example in transportation networks, whenever there is a delay, the transient determines the worst-case recovery time. Moreover in the case of link reversal routing (Gafni and Bertsekas, 1987), it is equal to the time complexity of the routing algorithm. Hartmann and Arguelles (1999) established an upper bound on the length of the transient part of general MPL systems via graph-theoretical techniques. Under the assumption of integer delays Charron-Bost et al. (2013) employed algebraical approaches to obtain an upper bound. In this work, we show how backward reachability analysis can be used to determine the length of the transient part of a model (given via its system matrix), for any desired initial state: this generalizes related results in the literature. The set of final conditions for this backward reachability problem is defined as the set of states with zero length of the transient part, namely the states for which the periodic behavior occurs immediately.

The article is structured as follows. Section 2 introduces models and notions needed to tackle the problem at hand. Section 3 discusses the procedure for backward reachability analysis. Section 4 describes applications of backward reachability in safety and transient analysis. Finally, Section 5 presents conclusions and discusses future work.

2. MODELS AND PRELIMINARIES

2.1 Max-Plus-Linear Systems

Define $\mathbb{R}_\varepsilon$, ε, and e respectively as $\mathbb{R} \cup \{\varepsilon\}$, $-\infty$, and 0. For $\alpha, \beta \in \mathbb{R}_\varepsilon$, introduce the two operations:

$$\alpha \oplus \beta = \max\{\alpha, \beta\} \qquad \text{and} \qquad \alpha \otimes \beta = \alpha + \beta,$$

where the element ε is considered to be absorbing w.r.t. $\otimes$ (Baccelli et al., 1992, Definition 3.4). Given $\beta \in \mathbb{R}$, the max-algebraic power of $\alpha \in \mathbb{R}$ is denoted by $\alpha^{\otimes \beta}$ and corresponds to $\alpha\beta$ in the conventional algebra. The rules for the order of evaluation of the max-algebraic operators correspond to those of conventional algebra: max-algebraic power has the highest priority, and max-algebraic multiplication has a higher precedence than max-algebraic addition (Baccelli et al., 1992, Sec. 3.1).

The basic max-algebraic operations are extended to matrices as follows. If $A, B \in \mathbb{R}_\varepsilon^{m \times n}; C \in \mathbb{R}_\varepsilon^{n \times p}$; and $\alpha \in \mathbb{R}_\varepsilon$

$$[\alpha \otimes A](i,j) = \alpha \otimes A(i,j),$$
$$[A \oplus B](i,j) = A(i,j) \oplus B(i,j),$$
$$[A \otimes C](i,j) = \bigoplus_{k=1}^{n} A(i,k) \otimes C(k,j),$$

for all i, j. Notice the analogy between $\oplus$, $\otimes$ and $+$, $\times$ for matrix and vector operations in conventional algebra. Given $m \in \mathbb{N}$, the m-th max-algebraic power of $A \in \mathbb{R}_\varepsilon^{n \times n}$ is denoted by $A^{\otimes m}$ and corresponds to $A \otimes \cdots \otimes A$ (m times). Notice that $A^{\otimes 0}$ is an n-dimensional max-plus identity matrix, i.e. the diagonal and nondiagonal elements are e and ε, respectively. In this paper, the following notation is adopted for reasons of convenience. A vector with each component that is equal to 0 (resp., $-\infty$) is also denoted by e (resp., ε). Furthermore, for practical reasons, the state space is taken to be $\mathbb{R}^n$, which also implies that the system matrix A has to be row-finite (cf. Definition 1).

Definition 1. (Cuninghame-Green, 1979). A max-plus matrix is called regular (or row-finite) if it contains at least one element different from ε in each row. □

An autonomous MPL system (Baccelli et al., 1992, Remark 2.75) is defined as:

$$x(k) = A \otimes x(k-1),$$

where $A \in \mathbb{R}_\varepsilon^{n \times n}$, $x(k-1) = [x_1(k-1) \ldots x_n(k-1)]^T \in \mathbb{R}^n$ for $k \in \mathbb{N}$. The independent variable k denotes an increasing discrete-event counter, whereas the state variable x defines the (continuous) timing of the discrete events. Autonomous MPL systems are characterized by deterministic dynamics, namely they are not affected by exogenous inputs.

Related to matrix A is the notion of precedence (or communication) graph and of regular (or row-finite) matrix.

Definition 2. (Baccelli et al., 1992, p. 39). The precedence graph of $A \in \mathbb{R}_\varepsilon^{n \times n}$, denoted by $\mathcal{G}(A)$, is a weighted directed graph with vertices $1, \ldots, n$ and arc (j, i) with weight $A(i,j)$ for each $A(i,j) \neq \varepsilon$. □

Example 1. Consider the following autonomous MPL system from (Heidergott et al., 2006, Sec. 0.1), representing the scheduling of train departures from two connected stations $i = 1, 2$ ($x_i(k)$ denotes the time of the k-th departure from station i):

$$x(k) = \begin{bmatrix} 2 & 5 \\ 3 & 3 \end{bmatrix} \otimes x(k-1), \text{ or equivalently,}$$
$$\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \begin{bmatrix} \max\{2 + x_1(k-1), 5 + x_2(k-1)\} \\ \max\{3 + x_1(k-1), 3 + x_2(k-1)\} \end{bmatrix}. \quad (1)$$

Notice that A is a row-finite matrix. Its precedence graph is shown in Fig. 1. □

The notion of irreducible matrix, to be used shortly, can be given via that of precedence graph.

Definition 3. (Baccelli et al., 1992, Th. 2.14). A max-plus matrix $A \in \mathbb{R}_\varepsilon^{n \times n}$ is called irreducible if its precedence graph $\mathcal{G}(A)$ is strongly connected. □

Recall that a directed graph is strongly connected if for any pair of distinct vertices i, j of the graph, there exists a path from i to j (Baccelli et al., 1992, p. 37). From a max-algebraic perspective, a matrix A is irreducible if the non-diagonal elements of $\bigoplus_{k=1}^{n-1} A^{\otimes k}$ are finite (not equal to ε).