Commun Nonlinear Sci Numer Simulat 59 (2018) 235–254

Contents lists available at ScienceDirect

Commun Nonlinear Sci Numer Simulat

journal homepage: www.elsevier.com/locate/cnsns

Research paper

Group foliation of finite difference equations

Robert Thompson[a,*], Francis Valiquette[b]

[a] Department of Mathematics and Statistics, Carleton College, Northfield, MN 55057, USA
[b] Department of Mathematics, SUNY New Paltz, New Paltz, NY 12561, USA

ARTICLE INFO

Article history:
Received 5 January 2017
Revised 27 July 2017
Accepted 23 November 2017

MSC:
65L12
65Q10

Keywords:
Equivariant moving frames
Finite difference equations
Group foliation
Joint invariants
Lie group actions

ABSTRACT

Using the theory of equivariant moving frames, a group foliation method for invariant finite difference equations is developed. This method is analogous to the group foliation of differential equations and uses the symmetry group of the equation to decompose the solution process into two steps, called resolving and reconstruction. Our constructions are performed algorithmically and symbolically by making use of discrete recurrence relations among joint invariants. Applications to invariant finite difference equations that approximate differential equations are given.

1. Introduction

First introduced by Sophus Lie, [21], and further developed by Ernest Vessiot, [40], the method of group foliation, also called group splitting, or group stratification, is a general procedure for obtaining solutions of differential equations admitting a symmetry group. Modern treatments of this method appear in the book of Ovsiannikov, [33], and the work of Martina, Nutku, Sheftel and Winternitz, [25,27]. Recently, a formulation based on exterior differential systems was developed by Anderson, Fels and Pohjanpelto, [1,9,34]. Another formulation based on the theory of equivariant moving frames was proposed in [23] for finite-dimensional symmetry groups and extended to infinite-dimensional symmetry groups in [38]. In the present work, we adapt the constructions introduced in [23,38] to finite difference equations.

Given a strongly G-invariant finite difference equation (see Definition 2.5), the group foliation method uses the foliation of the solution space of the equation by its symmetry group to split the search for solutions into two steps: a *resolving step* and a *reconstruction step*. In the resolving step, the solution space of the equation is projected onto the leaves of the foliation, where, under typical regularity assumptions, the leaves of the foliation are parameterized by joint invariants (also called finite difference invariants). In applications, the projection is obtained by solving a system of equations consisting of the original equation written in terms of joint invariants together with the integrability conditions originating from the syzygies among the invariants. In Vessiot's terminology, these equations form the *resolving system*. As Ovsiannikov observed in [33], the resolving system may be easier to solve than the original equation since the symmetry group has reduced the

* Corresponding author.
E-mail addresses: rthompson@carleton.edu (R. Thompson), valiquef@newpaltz.edu (F. Valiquette).
URL: http://people.carleton.edu/~rthompson (R. Thompson), http://www2.newpaltz.edu/~valiquef (F. Valiquette)

https://doi.org/10.1016/j.cnsns.2017.11.027

size of the solution space. Given a solution to the resolving system, the reconstruction step then consists of solving a system of first order finite difference equations for the left moving frame of G, called the *reconstruction equations*. Solutions to the original finite difference equation are then obtained by acting on the solution of the resolving system by solutions of the reconstruction equations.

Using the theory of discrete equivariant moving frames, the resolving and reconstruction steps described above can be performed algorithmically and symbolically. After reviewing the concepts of finite difference equations and symmetry in Section 2, the basic moving frame constructions are introduced in Section 3. Since our emphasis is geared towards developing the group foliation method, we refer to [24] for some of the more subtle theoretical justifications of the discrete moving frame method. We note that our notation and terminology differs slightly from that used in [24].

In the differential setting, one of the fundamental results of the equivariant moving frame method is the derivation of recurrence relations relating the normalized differential invariants and their exterior derivatives. In Section 4, a discrete version of the recurrence relations is introduced, relating normalized joint invariants and their shifts. In analogy with the continuous theory, the discrete recurrence relations reveal the structure of the algebra of joint invariants, and these recurrence relations can be computed symbolically without knowing the expressions for the joint invariants, requiring only the expressions for the group action and the choice of a cross-section defining a moving frame. As a result, the group foliation algorithm introduced in Section 6 is completely symbolic in the sense that it does not require coordinate expressions for the moving frame or the joint invariants.

For invariant finite difference equations that approximate differential equations, the implementation of the group foliation method provides a new type of invariant numerical scheme. This naturally leads to new questions concerning the accuracy and stability of such schemes. As a preliminary investigation, we first consider in Section 7 the Schwarzian differential equation

$$\frac{y_{xxx}}{y_x} - \frac{3}{2}\left(\frac{y_{xx}}{y_x}\right)^2 = F(x),$$

which is invariant under the group of special linear fractional transformations

$$X = x, \quad Y = \frac{ay+b}{cy+d}, \quad ad - bc = 1,$$

and prove a discrete analogue of the Schwarz Theorem for an invariant discretization of this equation. Continuing this example, we perform in Section 8 a numerical simulation based on an exact solution of the Schwarz equation that admits vertical asymptotes. In accordance with other numerical simulations using symmetry-preserving schemes, [2,3,7], the group foliation scheme has no difficulty integrating beyond the vertical asymptotes. However, via the group foliation scheme, this unexpected behavior can be clearly explained, at least for the problem at hand. In light of our numerical simulation, we expect that further applications of the group foliation method to symmetry-preserving schemes might shed some light on the numerical properties of those schemes.

2. Preliminaries

Let M be an m-dimensional manifold with local coordinates $z = (z^1, \dots, z^m)$. Given a p-dimensional submanifold $S \subset M$ with $1 \le p < m$, let $z = z(s)$ be a local parametrization of S with independent variable $s = (s^1, \dots, s^p)$. For each integer $0 \le n \le \infty$, let $\mathrm{J}^{(n)} = \mathrm{J}^{(n)}(M, p)$ denote the n^{th} order submanifold jet bundle defined as the set of equivalence classes under the equivalence relation of n^{th} order contact, [28]. Local coordinates on $\mathrm{J}^{(n)}$ are given by

$$(s, z^{(n)}) = (s, \ \dots z^a_{s^B} \ \dots) \quad a = 1, \dots, m, \quad 0 \le \#B \le n,$$

where $z^{(n)}$ indicates the collection of submanifold jet coordinates $z^a_{s^B}$ representing the derivatives $\partial^k z^a/(\partial s^1)^{b^1} \cdots (\partial s^p)^{b^p}$, where $B = (b^1, \dots, b^p)$ is an ordered multi-index of order $\#B = k$ with nonnegative components $b^\nu \ge 0$.

In the discrete setting, the continuous variable $s = (s^1, \dots, s^p) \in \mathbb{R}^p$ is replaced by an integer multi-index

$$N = (n^1, \dots, n^p) \in \mathbb{Z}^p \subset \mathbb{R}^p.$$

For each $N \in \mathbb{Z}^p$, let

$$z_N = z(N)$$

denote a point on the submanifold $S \subset M$. A discrete counterpart to the submanifold jet space $\mathrm{J}^{(n)}$ is given by the n^{th} *order forward discrete jet space* $\mathrm{J}^{[n]}$ with coordinates

$$(N, z_N^{[n]}) = (N, \ \dots z_{N+K} \ \dots) \quad N \in \mathbb{Z}^p, \quad 0 \le \#K \le n, \tag{1}$$

where $z_N^{[n]}$ indicates the collection of points z_{N+K} with $K \in \mathbb{Z}^p_{\ge 0}$ a nonnegative integer multi-index of order at most n.

Remark 2.1. There are a multitude of ways to approximate the n^{th} order submanifold jet space $\mathrm{J}^{(n)}$. One can use forward, backward, or centered difference approximations, and in numerical applications one might consider more points for greater