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Research paper

# Conformal invariance and conserved quantities of mechanical system with unilateral constraints

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ABSTRACT

By distinguishing the different constraint cases, the whole course and piecewise conserved quantities, which deduced from conformal invariance of mechanical systems with unilateral constraints, are given. The determining equation of conformal invariance of the system is obtained. The sufficient and necessary conditions for the conformal invariance must be Lie symmetry of the system are given. The forms of conformal factors are obtained. An example is given to illustrate the results in this paper.



## 1. Introduction

The research on symmetries and conserved quantities of mechanical systems possesses important theoretical and practical significance. The well known Noether symmetry has broadly applications in mathematics, dynamics and physics [1–9], it always can lead to conserved quantities. In fact, it is also named variational symmetry [4]. Besides Noether symmetry, there are Lie symmetry, Mei symmetry, and so on [10–20]. Above symmetries are all basing on the Lie continuous transformation group. In 1997, Galiullin et al. [21] discussed the conformal invariance of Birkhoff system and deduced Noether conserved quantities from this symmetry. Mei et al. [22] extended the conformal invariance to generalized Birkhoff equations and gave the Noether conserved quantities. The key question is to find out the conformal factor to the conformal invariance of dynamics. Considerable progress has been made on the application of conformal invariance to mechanical systems in decades [23–28].

The unilateral constraints exist in many mechanical systems [29–33], the motions of these systems can be represented by a set of differential equations with unilateral constraints. In mathematics the unilateral constraint can be represented by inequalities. The symmetries and conserved quantities of mechanical system with unilateral constraints had been extensively investigated [34–38]. However, the conserved quantities they got are all whole course invariants. In fact, there is a transition from free motion to constraint motion for the mechanical systems with unilateral constraints, so the conserved quantities should be piecewise for these motions. In this paper, we will distinguish this case and give the whole course and piecewise conserved quantities for mechanical systems with unilateral constraints. These results will be helpful for the study of the dynamic behaviour of systems.

In the present paper, we study the conformal invariance and conserved quantities of mechanical system with unilateral constraints. We first give the Lagrange equation of mechanical system with unilateral constraints. Secondly, basing on the Lie

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point transformation group, we give the mathematical definition form of conformal invariance of system. Thirdly, the sufficient and necessary conditions are proposed to ensure that conformal invariance of mechanical with unilateral constraints is Lie symmetry. Fourthly, we give the propositions that conformal invariance leads to whole course and piecewise conserved quantities. Finally, we give an example to illustrate the application of the results.

## 2. The differential equations of motion of mechanical system with unilateral constraints

We consider a mechanical system whose configuration is determined by $n$ generalized coordinates $q_s (s = 1, \cdots, n)$, and the system subjects to $g$ unilateral ideal holonomic constraints

$$f_\beta(t, \boldsymbol{q}) \geq 0 \qquad (\beta = 1, \cdots, g) \tag{1}$$

The differential equation of motion of the system can be expressed as

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = Q_s + \lambda_\beta \frac{\partial f_\beta}{\partial q_s} \qquad (s = 1, \cdots, n) \tag{2}$$

For every $\beta$, we have the relations $f_\beta \geq 0, \lambda_\beta \geq 0, f_\beta \lambda_\beta = 0$. where $L$ is Lagrangian function of the mechanical system, and $Q_s$ are nonconservative forces, $\lambda_\beta$ are constraint multipliers. Because the constraints are unilateral, the variable of the velocity of the system is not continuous, so the Eq. (2) is not closed. In order to make the Eq. (2) close, we assume the constraint hypersurface is absolutely smooth and the collision is perfect elastic.

If the system is in the constraints, that is the constraint Eq. (1) equal to zero, we have $f_\beta(t, \boldsymbol{q}) = 0$. Suppose the system is nonsingular, i.e., $D = \mathtt{det}(\frac{\partial^2 L}{\partial \dot{q}_s \partial \dot{q}_k}) \neq 0$, we can work out the constraint multipliers as $\lambda_\beta = \lambda_\beta(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$. Consequently, The Eq. (2) can be expressed as explicit form

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = Q_s + \Lambda_s \qquad (s = 1, \cdots, n) \tag{3}$$

where $\Lambda_s = \Lambda_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) = \lambda_\beta(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) \frac{\partial f_\beta}{\partial q_s}$. From Eq. (3) we can have

$$F_s \equiv A_{jk}\ddot{q}_k + B_s(t, \dot{\boldsymbol{q}}) - Q_s - \Lambda_s = 0, \qquad (s, k = 1, \cdots, n) \tag{4}$$

where $A_{sk} = \frac{\partial^2 L}{\partial \dot{q}_s \partial \dot{q}_k}$, and $B_s = \frac{\partial^2 L}{\partial \dot{q}_s \partial q_k}\dot{q}_k + \frac{\partial^2 L}{\partial \dot{q}_s \partial t} - \frac{\partial L}{\partial q_s}$. Expanding Eq. (3), we can get all the generalized acceleration

$$\ddot{q}_s = \alpha_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) \tag{5}$$

If the system is free from constraints, that is the constraint Eqs. (1) do not equal to zero, we have $f_\beta(t, \boldsymbol{q}) \neq 0$. then the Eq. (2) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = Q_s \qquad (s = 1, \cdots, n) \tag{6}$$

From Eq. (6) we can have

$$F_s^* \equiv A_{jk}\ddot{q}_k + B_s(t, \dot{\boldsymbol{q}}) - Q_s = 0, \qquad (s, k = 1, \cdots, n) \tag{7}$$

we can get all the generalized acceleration by expanding the Eq. (6)

$$\ddot{q}_s = \alpha_s^*(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) \tag{8}$$

We note that the Eq. (3) can only describe the motions of system accessing constraints without collision. In order to describe the collision course that the system comes into constraints, we can introduce the piecewise description or whole course description of the equations of motion of the system [39–41]. However, these descriptions can only treat with systems with one unilateral constraint. We will study the general collision course in the future work.

## 3. Conformal invariance of mechanical system with unilateral constraints

In order to get the conform invariance of mechanical system with unilateral constraints, we need to explore the transformation sets of independent and non-independent variables corresponding to Eqs. (1) and (2). We introduce the one-parameter Lie group of point transformations

$$\begin{aligned} t^* &= t + \varepsilon\xi_0(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \\ q_s^* &= q_s + \varepsilon\xi_s(s, \boldsymbol{q}, \dot{\boldsymbol{q}}), \ \ s = 1, \ldots, n \end{aligned} \tag{9}$$

where $\varepsilon$ is infinitesimal parameter, $\xi_0$, $\xi_j$ are infinitesimal transformation generators. It has infinitesimal generator vector

$$X^{(0)} = \xi_0 \frac{\partial}{\partial t} + \xi_s \frac{\partial}{\partial q_s} \tag{10}$$