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Research paper

Non-autonomous equations with unpredictable solutions

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ABSTRACT

To make research of chaos more amenable to investigating differential and discrete equations, we introduce the concepts of an unpredictable function and sequence. The topology of uniform convergence on compact sets is applied to define unpredictable functions [1,2]. The unpredictable sequence is defined as a specific unpredictable function on the set of integers. The definitions are convenient to be verified as solutions of differential and discrete equations. The topology is metrizable and easy for applications with integral operators. To demonstrate the effectiveness of the approach, the existence and uniqueness of the unpredictable solution for a delay differential equation are proved as well as for quasilinear discrete systems. As a corollary of the theorem, a similar assertion for a quasilinear ordinary differential equation is formulated. The results are demonstrated numerically, and an application to Hopfield neural networks is provided. In particular, Poincaré chaos near periodic orbits is observed. The completed research contributes to the theory of chaos as well as to the theory of differential and discrete equations, considering unpredictable solutions.

1. Introduction

From the applications point of view, the theory of differential and discrete equations focuses on equilibria, periodic, and almost periodic oscillations. They meet the needs of any real world problem related to mechanics, electronics, economics, biology, etc., if one searches for regular and stable dynamics of an isolated motion. However, they are not sufficient for many modern and prospective demands of robotics, computer techniques, and the internet, and chaotic dynamics comprise constructive properties for applications. This is the reason why it is important to join the power of deterministic chaos with the immensely rich source of methods for differential and discrete equations. We contributed to this in our studies [3–5] and the book [6], where a method of replication of chaos has been developed. It consists of the verification of ingredients of chaos such as sensitivity, transitivity, proximality and the existence of infinitely many unstable regular motions [7–9] for solutions of an equation with chaotic perturbation. This approach gives a very effective instrument for application of the accumulated knowledge in chaos research. Nevertheless, we are not glad with the necessity to check the presence of several ingredients. Therefore, in our opinion, unpredictable functions have become an instrument for the simplification of chaos analysis through differential and discrete equations.

In this paper, another step in the adaptation of unpredictable functions to the theory of differential equations has been made. We apply the uniform convergence on compact subsets of the real axis to determine unpredictable functions for two reasons. The first reason is that the topology is easily metrizable, in particular, to the metric for Bebutov dynamical

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system [10], and consequently, the unpredictable functions and solutions immediately imply the presence of Poincaré chaos according to our results in [11]. The second one is the easy verification of the convergence. Thus, the present study is useful for the theory of differential equations as well as chaos researches. For the construction of unpredictable functions we have applied the results on the equivalence of discrete dynamics obtained in papers [12,13]. Moreover, an application to Hopfield neural networks [14,15] is provided.

In the instrumental sense, discreteness has been the main object in chaos investigation. To check this, it is sufficient to recall the definitions of chaos [7–9], which are based on sequences and maps, as well as Smale Horseshoe and logistic maps, Bernoulli shift [9], which are in the core of the chaos theory. One can say that stroboscopic observation of a motion was the single way to indicate the irregularity in continuous dynamics. The definitions of chaos for continuous dynamics, which are not related to discreteness [4,16–18], are requested for embedding the research to the theory of differential equations. The research as well as the origin of the chaos [19] gave us strong arguments for the development of motions in classical dynamical systems theory [20] by proceeding behind Poisson stable points to unpredictable points [11]. Then, the dynamics have been specified such that a function that is bounded on the real axis is an unpredictable point [1,2]. In the papers [1,2], we have demonstrated that unpredictable functions are easy to be analyzed as solutions of differential equations. This paradigm is not completed, if one does not consider discrete equations. Therefore, in the present paper we also deliver discrete analogues for unpredictable functions, calling them unpredictable sequences, and prove assertions on the existence and uniqueness of unpredictable solutions of discrete equations for the first time in the literature. The results can be useful for applications and theoretical analyses, in particular, for the modern development of computer technologies, software, and robotics [21,22].

2. Quasilinear differential equations

Let us introduce the following definition.

Definition 2.1. A uniformly continuous and bounded function $\vartheta : \mathbb{R} \to \mathbb{R}^m$ is unpredictable if there exist positive numbers ϵ_0, δ and sequences $\{t_n\}$, $\{u_n\}$ both of which diverge to infinity such that $\|\vartheta(t+t_n) - \vartheta(t)\| \to 0$ as $n \to \infty$ uniformly on compact subsets of $\mathbb{R}$ and $\|\vartheta(t+t_n) - \vartheta(t)\| \geq \epsilon_0$ for each $t \in [u_n - \delta, u_n + \delta]$ and $n \in \mathbb{N}$.

To create Poincaré chaos [11], uniform continuity is not a necessary condition for an unpredictable function $\vartheta(t)$, and instead of the condition $\|\vartheta(t+t_n) - \vartheta(t)\| \geq \epsilon_0$ for each $t \in [u_n - \delta, u_n + \delta]$ and $n \in \mathbb{N}$, one can request that $\|\vartheta(t_n+u_n) - \vartheta(u_n)\| \geq \epsilon_0$ for each $n \in \mathbb{N}$. For the needs of verification of theorems on the existence of unpredictable solutions of differential equations we apply Definition 2.1, but for the future studies the following definitions may also be beneficial.

Definition 2.2. A continuous and bounded function $\vartheta : \mathbb{R} \to \mathbb{R}^m$ is unpredictable if there exist a positive number ϵ_0 and sequences $\{t_n\}$, $\{u_n\}$ both of which diverge to infinity such that $\|\vartheta(t+t_n) - \vartheta(t)\| \to 0$ as $n \to \infty$ uniformly on compact subsets of $\mathbb{R}$ and $\|\vartheta(t_n+u_n) - \vartheta(u_n)\| \geq \epsilon_0$ for each $n \in \mathbb{N}$.

Definition 2.3. A continuous and bounded function $\vartheta : \mathbb{R} \to \mathbb{R}^m$ is unpredictable if there exist a positive number ϵ_0 and sequences $\{t_n\}$, $\{u_n\}$ both of which diverge to infinity such that $\|\vartheta(t_n) - \vartheta(0)\| \to 0$ as $n \to \infty$ and $\|\vartheta(t_n+u_n) - \vartheta(u_n)\| \geq \epsilon_0$ for each $n \in \mathbb{N}$.

The main object of the present section is the following system of delay differential equations,

$$x'(t) = Ax(t) + f(x(t-\tau)) + g(t), \tag{2.1}$$

where τ is a positive number, the eigenvalues of the matrix $A \in \mathbb{R}^{m\times m}$ have negative real parts, $f : \mathbb{R}^m \to \mathbb{R}^m$ is a continuous function, and $g : \mathbb{R} \to \mathbb{R}^m$ is a uniformly continuous and bounded function. Our purpose is to prove that system (2.1) possesses a unique unpredictable solution which is uniformly exponentially stable, provided that the function $g(t)$ is unpredictable in accordance with Definition 2.1.

In the remaining parts of the paper, we will make use of the usual Euclidean norm for vectors and the norm induced by the Euclidean norm for square matrices.

Since the eigenvalues of the matrix A in system (2.1) have negative real parts, there exist numbers $K \geq 1$ and $\omega > 0$ such that $\left\|e^{At}\right\| \leq Ke^{-\omega t}$ for $t \geq 0$.

The following conditions are required.

- **(C1)** There exists a positive number M_f such that $\sup\limits_{x\in\mathbb{R}^m} \|f(x)\| \leq M_f$;
- **(C2)** There exists a positive number L_f such that $\|f(x_1) - f(x_2)\| \leq L_f\|x_1 - x_2\|$ for all $x_1, x_2 \in \mathbb{R}^m$;
- **(C3)** $\omega - 2\ KL_f e^{\omega\tau/2} > 0$.

The following theorem is concerned with the unpredictable solution of system (2.1).

Theorem 2.1. *Suppose that conditions* $(C1) - (C3)$ *are valid. If the function* $g(t)$ *is unpredictable, then system (2.1) possesses a unique uniformly exponentially stable unpredictable solution.*