Commun Nonlinear Sci Numer Simulat 000 (2017) 1–17

Contents lists available at ScienceDirect

Commun Nonlinear Sci Numer Simulat

journal homepage: www.elsevier.com/locate/cnsns

Research paper

Fractal attractors and singular invariant measures in two-sector growth models with random factor shares

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ARTICLE INFO

Article history:
Available online xxx

JEL classification:
C61
O41

Keywords:
Two-sector growth model
Stochastic factor shares
Fractal attractors
Singular measures

ABSTRACT

We analyze a multi-sector growth model subject to random shocks affecting the two sector-specific production functions twofold: the evolution of both productivity and factor shares is the result of such exogenous shocks. We determine the optimal dynamics via Euler–Lagrange equations, and show how these dynamics can be described in terms of an iterated function system with probability. We also provide conditions that imply the singularity of the invariant measure associated with the fractal attractor. Numerical examples show how specific parameter configurations might generate distorted copies of the Barnsley's fern attractor.

1. Introduction

Macroeconomic models, and in particular economic growth models, have attracted large interest over the last few decades because of their ability to generate complicated dynamics [5]. It is now well known that such models can also give rise to random dynamics eventually converging to invariant measures supported on fractal sets [23]. A growing number of studies has recently focused on characterizing the conditions under which this might occur by relying on the iterated function system (IFS) literature [1,10,30]. Most of the existing works dealing with economic growth and IFS analyze the traditional discrete time one-sector growth model with logarithmic utility and Cobb–Douglas production, in either its simplest setup or in slightly extended formulations; such a basic model through an appropriate transformation can be converted into a one-dimensional IFS, and it is thus possible to show that its optimal dynamic path may converge to a singular measure supported on a Cantor set, and also that the invariant probability may be either singular or absolutely continuous according to specific parameter configurations [16,19–23,26]. Very few are those that instead consider more sophisticated two-sector growth models giving rise to a two-dimensional IFS; the analysis in this framework is clearly more complicated but it is still possible to show that the optimal dynamic path may converge to a singular measure supported on some fractal set, like the Sierpinski gasket, and to eventually characterize singularity versus absolute continuity of the invariant probability [12,13].

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http://dx.doi.org/10.1016/j.cnsns.2017.07.008
1007-5704/

We wish to contribute to this literature by extending the analysis of two-sector random growth models and their relation with fractal steady states in order to allow for the random shock to affect not only the productivity level of the sector-specific production functions [12,13], but also their factor shares. To the best of our knowledge, the possibility of exogenous shocks on factor shares thus far have been considered only in the one-sector growth model by Mirman and Zilcha [18], which has recently been extended to the case of learning by Mirman et al. [17]. Nonetheless, it is an interesting generalization of the traditional setup both from the economic and mathematical point of view; indeed, variable factor shares may describe the change in the structure of economic activities which we have observed in industrialized economies over the last decades [15,24], and also imply that the optimal economic dynamics may be characterized by an IFS with variable coefficients which makes the analysis of convergence and invariant probability properties not trivial at all. In order to look at this in the simplest possible setup we build on the model discussed in [12] in which endogenous growth is ruled out (see [13], for a discussion of how results may differ in a framework with endogenous growth), and show that through an appropriate log-transformation the optimal nonlinear dynamic system can be converted into a topologically equivalent linear IFS, although such a transformation requires us to impose a substantial number of restrictions on the model's parameters. We can however show that the system converges to a singular measure supported on some fractal set, which (because of the imposed restrictions) turns out to be a distorted copy of Barnley's fern. We also provide some sufficient conditions under which the associated self-similar measure may be singular.

The paper proceeds as follows. In Section 2 we briefly summarize some basic results from the IFS theory and present some novel sufficient conditions (Theorem 1) for testing the singularity of the invariant distribution in a two-dimensional IFS setup. In Section 3 we analyze a two-sector economic growth model in which random shocks affect both the productivity and the factor shares of the two sector-specific production functions, and we fully characterize the optimal policies through Euler–Lagrange equations. In Section 4 we introduce a log-transformation which allows us to reduce the nonlinear IFS associated with the optimal dynamics to a topologically equivalent linear IFS, which substantially simplifies our analysis but requires to impose some restrictions on the possible parameter values. We also provide, sufficient conditions for the attractor of this linear IFS to be a fractal set (the Barnsley's fern), and we identify sufficient conditions under which the self-similar measure may turn out to be singular. Section 5 presents specific examples of attractor and in particular it shows that the parameter restrictions required by our log-transformation preclude us from generating the original fern, and thus we can obtain only distorted copies of it. We also show that by relying on specific parameters values consistent with empirical evidence, the degree of distortion substantially increases and the attractor does no longer resemble a fern. Section 6 presents concluding remarks and proposes directions for future research. The proofs of the main propositions and theorems are presented in the Appendix.

2. Iterated function systems and fractal attractors

Hutchinson [10] and, shortly thereafter, Barnsley [1] showed how systems of contractive maps with associated probabilities, referred to as IFS by the latter, can be used to construct fractal, self-similar sets and measures. More in general, the action of a generalized fractal transform (GFT) [11] $T\colon X\to X$ on an element u of the complete metric space (X, d) can be summarized in the following steps. It produces a set of N spatially-contracted copies of u and then it modifies the values of these copies by means of a suitable range-mapping. Finally, it recombines them using an appropriate operator in order to get the element $v\in X$, $v=Tu$. In all these cases, under appropriate conditions, the fractal transform T is a contraction and thus Banach's fixed point theorem guarantees the existence of a unique fixed point $\bar{u}=T\bar{u}$. Furthermore the fixed point $\bar{u}$ is continuous with respect to perturbations of the operator T in the d_∞ distance (see [11]), meaning that if T_1 and T_2 are two contractions with contractivities c_1, c_2 and fixed points $\bar{u}_1$ and $\bar{u}_2$, then

$$d(\bar{u}_1,\bar{u}_2)\le\frac{1}{1-\min\{c_1,c_2\}}\sup_{u\in X}d(T_1u,T_2u) \tag{1}$$

The inverse problem is a key factor for applications: given a “target” element $v\in X$, we look for a point-to-point contraction mapping T with fixed point $\bar{u}$ such that $d(v,\bar{u})$ is as small as possible. In practical applications, however, it is difficult to construct solutions to this problem and we generally rely on the following simple consequence of Banach's fixed point theorem, known in the fractal coding literature as the *collage theorem*, which states that

$$d(v,\bar{u})\le\frac{1}{1-c}d(v,Tv)$$

(c is the contractivity factor of T). Instead of trying to minimize the error $d(v,\bar{u})$, we look for a contraction mapping T that minimizes the *collage error* $d(v, Tv)$.

2.1. Self-similar attractors and invariant measures

An *N-map iterated function system* (IFS) ([1,10]) is a set of N contraction maps $w_i\colon X\to X$, *i.e.*, for each $1\le i\le N$, there exists a $c_i\in[0,1)$ such that $d(w_i(x), w_i(y)\le c_i d(x, y)$ for all $x, y\in X$. Associated with an N-map IFS is a set-valued mapping