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Research paper

Coupled chaotic fluctuations in a model of international trade and innovation: Some preliminary results

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ABSTRACT

We consider a two-dimensional continuous noninvertible piecewise smooth map, which characterizes the dynamics of innovation activities in the two-country model of trade and product innovation proposed in [7]. This two-dimensional map can be viewed as a coupling of two one-dimensional skew tent maps, each of which characterizes the innovation dynamics in each country in the absence of trade, and the coupling parameter depends inversely on the trade cost between the two countries. Hence, this model offers a laboratory for studying how a decline in the trade cost, or globalization, might synchronize endogenous fluctuations of innovation activities in the two countries. In this paper, we focus on the bifurcation scenarios, how the phase portrait of the two-dimensional map changes with a gradual decline of the trade cost, leading to border collision, merging, expansion and final bifurcations of the coexisting chaotic attractors. An example of peculiar border collision bifurcation leading to an increase of dimension of the chaotic attractor is also presented.

1. Introduction

To investigate how globalization might affect the co-movement of innovation activities across countries, Matsuyama et al. [7] developed a two-country model of trade and product innovation, and showed that the innovation dynamics of this two-country model is characterized by a *two-dimensional* (2D for short) *continuous noninvertible piecewise* map: F: $(x, y) \mapsto F(x, y)$, where $x > 0$ and $y > 0$ measure the variety of products that have been innovated in the past and continue to be produced in each country. This map has four parameters, one of which, $\rho \in [0, 1)$, is an inverse measure of the trade cost between the two countries. At a prohibitively high trade cost, the two countries are isolated from each other and $\rho = 0$. In this case, the dynamics of innovation in each country are independent of each other and characterized by a *1D skew tent map* (i.e., a continuous, noninvertible piecewise linear map: see, e.g., [5,6,13]). Furthermore, for the permissible values of the three remaining parameters of this model, each of these two decoupled skew tent maps has a unique attracting fixed point, an attracting period 2 cycle or a 2^i-cyclic chaotic attractor ($i \geq 0$). In this model, a gradual decline in the trade cost causes a gradual market integration of the two countries, or globalization, which is captured by a gradual increase in ρ, leading to a coupling of innovation dynamics in the two countries. Thus, this 2D map offers an ideal laboratory for studying

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how globalization affects the co-movement of innovation activities across countries. In [7], the analysis of this 2D map was restricted to the case where each of the two decoupled 1D skew tent maps has an attracting cycle of period 2. In this case, the 2D map has at most two coexisting attracting cycles of period 2, "the synchronized 2-cycle" and "the asynchronized 2-cycle". Along the synchronized 2-cycle, product innovation is active and inactive at the same time in the two countries, while it is active only in one country along the asynchronized 2-cycle. It was shown that a gradual increase in ρ causes the basin of attraction of the synchronized 2-cycle to expand and the basin of attraction of the asynchronized 2-cycle to shrink, and that there exists a critical value $\rho_c < 1$, such that the asynchronized 2-cycle is unstable and the synchronized 2-cycle is the only attractor of this 2D system, for $\rho \in (\rho_c, 1)$. Thus, even a partial market integration would cause a full synchronization. Furthermore, it was shown that this critical value ρ_c is lower when the two countries are more unequal in size, which means that a smaller reduction in the trade cost would cause a full synchronization of the innovation activities across the two countries of more unequal size.

In the present paper we continue our investigation of this 2D map by examining the bifurcation scenarios caused by an increase in ρ, including the case where each of the two decoupled 1D skew tent maps has a 2^i-cyclic chaotic attractor ($i \geq 0$). Thus, the decoupled 2D map has 2^i coexisting 2^i-cyclic chaotic attractors, and for increasing ρ some of these attractors disappear and some new attractors appear. In particular, we present examples of so-called *expansion bifurcation* leading to an abrupt increase in the size of a chaotic attractor, *merging bifurcation* associated with direct pairwise merging of the pieces of a chaotic attractor, and *final bifurcation* related to the transformation of a chaotic attractor into a chaotic repellor. As discussed in [1], such bifurcations are caused by a contact of the attractor with its immediate basin boundary which can be regular or fractal. Due to *nonsmoothness* of map F which is defined by four smooth maps in four subregions of the phase plane, so-called *border collision bifurcation* (BCB for short) can also be involved into the bifurcation sequences. Recall that a BCB occurs when the phase portrait of a piecewise smooth map changes qualitatively due to a contact of an invariant set with a border, often called *switching manifold*, along which the system function changes its definition (see [11,13]). The four switching manifolds of the considered map F certainly increase the number of various outcomes of a BCB. In particular, we present an example of a chaotic attractor born due to a BCB, as well as an example of BCB which leads to an increase of dimension of the colliding attractor, namely, cyclic chaotic intervals bifurcate into a cyclic chaotic attractor with Cantor-like structure.

One more important property of the considered map F is its *noninvertibility*. A powerful tool for the investigation of the dynamics of a noninvertible map is the *theory of critical lines* developed in [8] (see also [10]). Recall that a critical line of a 2D continuous noninvertible map is defined as the locus of points having at least two coincident rank-1 preimages. For a 2D *smooth* noninvertible map an image of a set, associated with a vanishing Jacobian determinant, may possess such a property. The considered *piecewise smooth* noninvertible map F has four critical lines each of which is an image of the related switching manifold. We show how these critical lines and their images are used to determine boundaries of chaotic attractors[1] of map F. Critical lines may also be responsible for several interesting transformations of basins of attraction. For example, due to a contact of a basin with a critical line new islands of this basin may appear inside the basin of some other attractor. Similar transformations are described, e.g. in [9]. Recall that in continuous invertible maps the basins of attraction are necessarily simply connected sets, while attractors of noninvertible maps may have connected but not simply connected, or disconnected basins, whose occurrence is related to contact bifurcations with critical lines.

Discussing chaotic attractors of map F we use more general concepts of synchronized and asynchronized fluctuations comparing with those used in [7] for the attracting 2-cycles. Namely, chaotic innovation fluctuations are called synchronized if one observes simultaneous increase or decrease of the values of both variables along the trajectory, and fluctuations are asynchronized if an increase/decrease of the value of one variable is accompanied by a decrease/increase of the value of the other variable.

The paper is organized as follows. In Section 2 we formally introduce map F, briefly explain economics behind it, and discuss its simplest properties. In Section 3 the results related to the dynamics of the skew tent map are applied to the map F for $\rho = 0$. Considering these results as a starting point, in Section 4 we present several bifurcation sequences associated with chaotic attractors, which are observed in the coupled map F when the value of ρ is gradually increased. In case of the countries of equal size discussed in Section 4.1, map F is symmetric with respect to the main diagonal, so that any invariant set S (e.g., an attractor) of F is either symmetric itself, or there exist one more invariant set S' which is symmetric to S. As a results, the bifurcations of coexisting chaotic attractors, which are symmetric to each other with respect to the diagonal, occur simultaneously. In contrast, in the case of the two countries of unequal size, discussed in Section 4.2, map F is asymmetric, and the bifurcations of coexisting chaotic attractors do not occur at the same parameters values, leading to a richer bifurcation scenarios. Section 5 concludes.

[1] Note that chaotic attractors of 2D *invertible* maps, such as, for example, the well known Henon attractor, have a Cantor-like structure, while 2D *noninvertible* maps can have also full measure chaotic attractors.

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