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Research paper

# Bifurcation and nonlinear analysis of a time-delayed thermoacoustic system

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ABSTRACT

In this paper, of primary concern is a time-delayed thermoacoustic system, viz. a horizontal Rijke tube. A continuation approach is employed to capture the nonlinear behaviour inherent to the system. Unlike the conventional approach by the Galerkin method, a dynamic system is naturally built up by discretizing the acoustic momentum and energy equations incorporating appropriate boundary conditions using a finite difference method. In addition, the interaction of Rijke tube velocity with oscillatory heat release is modeled using a modified form of King's law. A comparison of the numerical results with experimental data and the calculations reported reveals that the current approach can yield very good predictions. Moreover, subcritical Hopf bifurcations and fold bifurcations are captured with the evolution of dimensionless heat release coefficient, generic damping coefficient and time delay. Linear stability boundary, nonlinear stability boundary, bistable region and limit cycles are thus determined to gain an understanding of the intrinsic nonlinear behaviours.



## 1. Introduction

Thermoacoustic instability widely exists in various combustion systems, such as propulsion systems, rocket motors, industrial burners, gas turbine engines etc. [1–3]. It arises from the interaction between the heat release and acoustic pressure or velocity oscillations within the combustion system. Under proper conditions, the acoustic perturbations would be strengthened and large amplitude limit cycles could occur. The energy density is usually so high that the resulting limit cycles would cause severe damages to the combustors. Hence, the oscillations have to be either avoided or controlled to an acceptable level. This requires a through and deep understanding of the instability mechanism, including the triggering, damping, flame-acoustic coupling etc. There has been a long history on the research of thermoacoustic instability. Following the very early observations of Sondhauss [4] and Rijke [5], Rayleigh defined the condition to trigger this instability, known as the Rayleigh criteria [6]. Thenceforth, this topic had long attracted the attention of researchers, especially with the development of rockets, jet engines and other industrial combustion systems. The pioneering theoretical work done by Cuclik [7,8] analyzed the nonlinear behaviour of acoustic waves within a combustion chamber and provided a formal framework to study the growth and limiting amplitude of acoustic waves. Dowling made tremendous contribution on the understanding of acoustic-combustion interaction [9,10], control of thermoacoustic instability [11] and application in aeronautic and power generation combustors [12]. Morgans and her co-workers [13,14] used a network model combined with flame describing function to predict the nonlinear thermoacoustic behaviour in combustors. Juniper [15] employed adjoint

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looping of the nonlinear governing equations as well as an optimization routine to study the triggering mechanism, including the non-normality, transient growth and bypass transition. Heckl [16] developed an empirical model for the nonlinear behaviour of both heat release and the reflection coefficients and this model was widely used to predict the limit cycle. Hoejmakers, Doey and Nijmeijer [17–19] utilized binary classification theory and network models to predict the stability of a thermoacoustic system. The intrinsic flame stability and flame-acoustic coupling were also investigated in real burners. Polifke and co-workers have also done much work on the instability study of thermoacoustic systems using Hybrid CFD/ low-order modelling [20], state-space models [21], adjoint Helmholtz solver [22], and frequency domain system model [23]. Poinsot and colleagues conducted large eddy simulation and experiments on the measurement of flame transfer function [24,25], evaluation of dynamic flame response [26,27] and acoustic analysis of thermoacoustic instability [28,29] in gas turbine combustion chambers. Candel [30–34] also explored the interactions between acoustics and swirling flames and came up with a unified framework for nonlinear combustion instability analysis. Sujith and his co-workers implemented analytical and experimental study and also bifurcation analysis to understand the dynamic behaviours involved in thermoacoustic systems, such as non-normality [35], nonlinearity [36], route to chaos [37], intermittency [38,39] etc. Yang has done numerous fundamental, numerical and experimental work on thermoacoustic instability encountered in gas turbine [40], liquid rocket engines [41] and liquid-fuelled propulsion systems [42]. Campa and Camporeale [43–45] used Finite Element Method to predict the acoustically driven combustion instabilities and investigated the influence of flame and burner transfer matrix on the instability mode and frequencies. More work in this field can be found in review papers [46–49].

Since the instability is generally not desirable, it has to be controlled or avoided. There have been many passive or active control methods, such as Helmholtz resonator [50,51], perforated liner [52], jet injection [53], fuel injection [54] and acoustic feedback [3]. However, these methods are still not adequate enough to be applied into systems like jet engines because of either inefficiency or the lack of suitable actuators for operation in such harsh working environments. Thus, an alternative approach is to avoid the thermoacoustic oscillations by defining a safe operation region. Within this region, the oscillations would either not occur or retain at sufficiently low amplitudes. Due to the strong nonlinear characteristics, the system behaviours are generally functions of operation parameters. Industrial combustors are often experimentally tested under different working conditions to find the safe operation region, which is extremely expensive and complicated. Therefore, it would be a good choice to computationally estimate the safe operation region, which thus raises the demand to come up with accurate numerical methods. Since practical combustors normally involve complex geometries, turbulent flow and combustion, it would be a good starting point to investigate the numerical method in a simple thermoacoustic system. Rijke tube has been a classical tool to study the thermoacoustic instability. It usually consists of an open-end tube and heat source inside it. When the heat source is placed in certain positions along the tube, sound would emit from the tube. The sound is generated due to the transfer from unsteady heat release to acoustic energy. Despite the simplicity in structure, it contains rich nonlinear behaviours, such as bifurcation, limit cycle, quasiperiodicity and chaos, which make it an excellent example for the study of thermoacoustic instability [11, 55,56].

In the past decades, Rijke tube has been extensively studied to understand the intrinsic nonlinear behaviour of the thermoacoustic instability. Hantschk and Vortmeyer [57] investigated self-excited thermoacoustic instabilities in the Rijke tube using a commercial CFD code. Two different kinds of Rijke tubes were modelled and the results showed good agreement with experiments. Non-linearity in the heat flux from the heating source to the flow was found to determine the limit cycle amplitudes. Matveev [58,59] combined linear theory and thermal analysis to predict the linear stability boundaries in a horizontal Rijke tube. A special form of the nonlinear heat transfer function was introduced to extend the method to nonlinear stability analysis. Hysteresis phenomenon was reported in the stability boundary and limit cycles were predicted as observed in experiments. Ananthkrishnan et al. [60] obtained the reduced-order models to capture the global behaviour of chamber dynamics via truncating the modal expansions and determined the number of modes required for accurate results. Heckl and Howe [61] conducted stability analysis of the Rijke tube by making use of a Green's function. Oscillations were described in terms of the eigenmodes of an integral equation derived using the Green's function and the predictions of stability behaviour were in line with Rayleigh's criterion. Balasubramanian and Sujith [35] studied the role of non-normality and nonlinearity in thermoacoustic system in a Rijke tube using the heat release model from Heckl [16]. It was shown that the non-normality inherent in the thermoacoutic system could result in transient growth of oscillations which can trigger nonlinearities in the system. Subramanian et al. [62] conducted bifurcation analysis of the dynamic behaviours of a horizontal Rijke tube and obtained bifurcation plots as a function of different system parameters. No/linear stability boundaries and other nonlinear phenomena were also observed in the analysis of the thermoacoustic system. Noble et al. [63] described a data-driven nonlinear and chaos theory–based analysis of thermoacoustic instabilities in a simple Rijke tube. It only relied on experimental data with no implicit assumptions. PLIFH measurement of OH radical at the rate of 2500 Hz was used to capture the thermoacoustic instability modes appeared in the Rijke tube. Chaotic behaviour was identified in the thermoacoustic instability.

In order to study thermoacoustic instability, most models adopt the conservation equations for mass, momentum and energy to represent the nature of thermoacoustic system. These equations are in the form of partial differential equations (PDEs) and need to be discretized and solved using proper numerical methods. Galerkin method [64] has been the most commonly employed technique in previous research. Nevertheless, in present study, a novel approach named Method of Lines (MOL) [65] was employed to convert the governing PDEs into a series of to ordinary differential equations (ODEs). It has seldom been used before in this field owing to the complexity of discretization and expensive computational cost. As shown in later sections, good predictions can be gained in comparison with the conventional method. Most importantly,