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Research paper

A generalized super AKNS hierarchy associated with Lie superalgebra $sl(2|1)$ and its super bi-Hamiltonian structure

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ABSTRACT

Starting from a 3×3 matrix-valued spectral problem associated with a Lie superalgebra $sl(2|1)$, a generalized super Ablowitz–Kaup–Newell–Segur (AKNS) hierarchy is derived. The resulting super AKNS hierarchy has a super bi-Hamiltonian structure by the supertrace identity.

1. Introduction

Searching for new soliton hierarchies plays an important role in the soliton and integrable systems. Matrix spectral problem or Lax pair is a crucial key to construct soliton hierarchies. And the trace identity provides a powerful method to construct Hamiltonian structures of the resulting soliton hierarchies. In what follows, let us recall the standard procedure for constructing soliton hierarchies. Suppose a given spatial spectral problem as

$$\phi_x = U\phi, \quad U = U(u, \lambda) \in \tilde{g}, \tag{1}$$

where $\tilde{g}$ is a matrix loop algebra, u is a potential and λ is a spectral parameter. We solve the stationary equation

$$V_x = [U, V],$$

where

$$V = V(u, \lambda) = \sum_{i \geq 0} V_i \lambda^{-i}, \quad V_i \in \tilde{g}, \quad i \geq 0.$$

Then, we formulate the temporal spectral problems:

$$\phi_{t_n} = V^{(n)}\phi = V^{(n)}(u, \lambda)\phi, \tag{2}$$

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where

$$V^{(n)}=(\lambda^n V)_+ + \Delta_n, \quad \Delta_n \in \tilde{g}, \quad n \geq 0,$$

with P_+ means the polynomial part of P in λ. The compatibility conditions of (1) and (2), i. e. the zero curvature equations, are given by

$$U_{t_n} - V_x^{(n)} + [U, V^{(n)}] = 0, \quad n \geq 0, \tag{3}$$

which will engender a hierarchy of soliton equations

$$u_{t_n} = K_n(u), \quad n \geq 0. \tag{4}$$

With the aid of the trace identity [1]:

$$\frac{\delta}{\delta u}\int tr\left(\frac{\partial U}{\partial \lambda}V\right)dx = \lambda^{-s}\frac{\partial}{\partial \lambda}\lambda^s tr\left(\frac{\partial U}{\partial u}V\right), \tag{5}$$

the obtained soliton hierarchy (4) has the following Hamiltonian structure:

$$u_{t_n} = K_n(u) = J\frac{\delta H_n}{\delta u}, \quad n \geq 0, \tag{6}$$

where J is a Hamiltonian operator and all of the H_n are Hamiltonian functions. The Hamiltonian structures of some famous soliton hierarchies (such as the AKNS hierarchy [2], the Kaup–Newell (KN) hierarchy [3], the Wadati–Konno–Ichikawa (WKI) hierarchy [4], the Boiti–Pempinelli–Tu (BPT) hierarchy [5], and so on) are constructed in Ref. [1].

This method is usually called the Tu scheme, which has successfully applied to the super spectral problems. Successful examples include the super AKNS hierarchy [6,7], the super Dirac hierarchy [6,8], the super coupled Korteweg–de Vires (cKdV) hierarchy [9], the super KN hierarchy [10,11], etc. [12–14]. And in these references, their super Hamiltonian structures are respectively furnished by the supertrace identity.

In recent years, generalized hierarchies of the classical soliton equations have been widely studied by many researchers. For example, the generalized AKNS hierarchy [15–17], the generalized KN hierarchy [18], the generalized WKI hierarchy [19] and so forth. In very recent years, Grahovski and Mikhailov proposed a new super soliton equation (s-cNLS) with two boson variables and two fermi variables [20]. Zhou has showed that the s-cNLS equation is actually a member of the $sl(2|1)$ super AKNS hierarchy [21]. Here we shall consider a generalization of the super AKNS hierarchy related to a Lie superalgebra $sl(2|1)$.

The paper is organized as follows. In the next section, we shall derive a generalized super AKNS hierarchy associated with a Lie superalgebra $sl(2|1)$. Then in Section 3, the resulting generalized super AKNS hierarchy can be written as the super bi-Hamiltonian structure by making use of the supertrace identity. Some conclusions and discussions are listed in the last section.

2. A generalized *sl*(2|1) super AKNS hierarchy

Let us start with the following matrix-valued spectral problem associated with a Lie superalgebra $sl(2|1)$:

$$\phi_x = U(u,\lambda)\phi, \quad U(u,\lambda)=\begin{pmatrix} \lambda + r & p & \alpha \\ q & -\lambda - r & \beta \\ \gamma & \zeta & 0 \end{pmatrix}, \quad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}, \tag{7}$$

where $u=(p,q,\alpha,\beta,\gamma,\zeta)^T$ is a potential, λ is a spectral parameter, p and q are even potentials, and α, β, γ and ζ are odd potentials. Here we note that $r=\varepsilon(pq+\alpha\gamma+\beta\zeta)$ with ε is an arbitrary even constant.

Remark 1.

(1) When $\varepsilon = 0$ in (7), the spectral matrix U is exactly one of the spectral problem (1) in Ref. [21] by proper variable substitutions.
(2) When $\varepsilon = 0$, $\gamma = \beta$ and $\zeta = -\alpha$ in (7), the spectral matrix U is exactly one of the super AKNS case. For detail, we can refer to the references [6,7,22–24].

Therefore, the spatial spectral problem (7) is regarded as an extension of the $sl(2|1)$ super AKNS spectral problem. To derive the generalized hierarchies of equations, we solve the stationary zero curvature equation

$$V_x = [U, V], \tag{8}$$

where

$$V=\begin{pmatrix} A & B & \rho \\ C & E-A & \delta \\ \xi & \eta & E \end{pmatrix}, \tag{9}$$