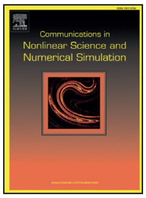
Commun Nonlinear Sci Numer Simulat 44 (2017) 414–423

Contents lists available at ScienceDirect

Commun Nonlinear Sci Numer Simulat

journal homepage: www.elsevier.com/locate/cnsns



Research paper

Probabilistic solutions of nonlinear oscillators excited by combined colored and white noise excitations

Guo Siu-Siu, Shi Qingxuan*

School of Civil Engineering, Xi'an University of Architecture and Technology, Shaanxi, 710055, China

ARTICLE INFO

Article history:
Received 23 November 2015
Revised 2 September 2016
Accepted 2 September 2016
Available online 4 September 2016

Keywords:
Fokker-Planck-Kolmogorov (FPK) equation
Colored noise
Filtered normal process
Narrow-banded noise

ABSTRACT

In this paper, single-degree-of-freedom (SDOF) systems combined to Gaussian white noise and Gaussian/non-Gaussian colored noise excitations are investigated. By expressing colored noise excitation as a second-order filtered white noise process and introducing colored noise as an additional state variable, the equation of motion for SDOF system under colored noise is then transferred artificially to multi-degree-of-freedom (MDOF) system under white noise excitations with four-coupled first-order differential equations. As a consequence, corresponding Fokker-Planck-Kolmogorov (FPK) equation governing the joint probabilistic density function (PDF) of state variables increases to 4-dimension (4-D). Solution procedure and computer programme become much more sophisticated. The exponential-polynomial closure (EPC) method, widely applied for cases of SDOF systems under white noise excitations, is developed and improved for cases of systems under colored noise excitations and for solving the complex 4-D FPK equation. On the other hand, Monte Carlo simulation (MCS) method is performed to test the approximate EPC solutions. Two examples associated with Gaussian and non-Gaussian colored noise excitations are considered. Corresponding band-limited power spectral densities (PSDs) for colored noise excitations are separately given. Numerical studies show that the developed EPC method provides relatively accurate estimates of the stationary probabilistic solutions, especially the ones in the tail regions of the PDFs. Moreover, statistical parameter of mean-up crossing rate (MCR) is taken into account, which is important for reliability and failure analysis. Hopefully, our present work could provide insights into the investigation of structures under random loadings.

1. Introduction

It is known that some noises in the real world, such as seismic ground motion [1], wind velocity, sea wave, or noise in biological systems [2,3], are colored or non-white noise processes. These physical processes are generally characterized by correlation functions with finite correlation length, and are clearly different from white noise processes with zero correlation length. Alternatively, these colored or non-white noise excitations can be expressed as filtered white noise processes and modeled as responses of the first- or second-order linear or nonlinear oscillators excited by Gaussian white noise excitations [4]. One of the simplest examples of finite time correlation noise is the Ornstein-Uhlenbeck process, $C(t)$, with an exponential

---

* Corresponding author.
E-mail address: shiqx@xauat.edu.cn (S. Qingxuan).

http://dx.doi.org/10.1016/j.cnsns.2016.09.003
1007-5704/© 2016 Elsevier B.V. All rights reserved.

correlation function of the form [5,6]

$$E[C(t)C(s)] = \frac{D}{\tau}\exp\left[-\frac{|t-s|}{\tau}\right], \tag{1}$$

where $E[\cdot]$ is the expectation operator, τ denotes the correlation time, D is corresponding power spectral density (PSD). Such exponentially correlated Gaussian process, C, can be obtained by passing Gaussian white noise $W(t)$ through a first-order low pass filter of the following form [7–9]

$$\dot{C}(t) = -\frac{1}{\tau_1}C + \frac{1}{\tau_1}W(t). \tag{2}$$

This equation is a linear filter and hence the filtered white noise is also Gaussian. Non-Gaussian process can be obtained by filtering white noise through a nonlinear filter.

Excited by colored noise, state variables of systems are not Markov process. Fokker-Planck-Kolmogorov (FPK) equation cannot be used directly to obtain probabilistic characters of system responses. However, if state variables of the interested system and colored noise are assumed to be initially uncorrelated and colored noise is introduced as an auxiliary state variable, then the joint response process consisting of state variables and colored noise is Markovian. Hence analytical methods for Markov process are appropriate [10,11]. However, this mathematical artefact augments the dimensionality of state-space and gives rise to four-dimensional (4-D) FPK equation. It seems to increase the complexity of problem to a greater degree since it is known that solving multi-dimensional FPK equation is a challenge. Corresponding solution procedure is much more sophisticated.

So far, exact probability density function (PDF) solutions have been available merely to two-dimensional FPK equations on a small scale. Many approximate methods and numerical methods are also just effective for two-dimensional FPK equations. Only a few approximate or numerical methods can be used for multi-dimensional FPK equations. Equivalent linearization (EQL) method, proposed by Booton on the control of electronic systems, was broadly adopted for analyzing weakly nonlinear systems under external excitations [12,13]. This method is based on the assumption that system responses are Gaussian. Hence the first and second moments of system responses can well characterize the interested systems. Furthermore, when the nonlinearity becomes strong or when there is multiplicative excitation, moment closure method or cumulant-neglect closure method, as an generalization of EQL method, is developed to approximate higher-order statistical moments or cumulants [14–16]. The EQL or generalized EQL method has already been employed for multi-dimensional systems, and it can definitely be used to 4-D FPK equations for systems excited by colored noise. But statistical moments obtained with these methods cannot completely characterize probabilistic properties of state variables. With the help of information on statistical moments of system responses, exponential polynomial closure (EPC) method, broadly applied for cases of single-degree-of-freedom (SDOF) systems under white noise excitations, can be facilitated to solve two-dimensional FPK equations. It has been verified effective in many nonlinear systems of polynomial type [17–19]. Furthermore, it is considered as the foundation for analyzing multi-degree-of-freedom (MDOF) systems with split-state-space-EPC (3S-EPC) method [20,21]. In this paper, the EPC method is further improved and developed for the case of system under combined white and colored noises and for solving 4-D FPK equation. Much more complex computer program and more computational effort are required. Numerical method Monte Carlo simulation (MCS) method has often been used to test the approximate solutions [22,23]. The MCS can be employed for analyzing multi-dimensional systems, but the computational effort is huge, particular in the cases of strong nonlinearity or small probability events.

The focus of this paper is on the investigation of systems under combined white and colored noise excitations with the developed EPC method. Firstly, by introducing colored noise as an additional state variable, the second-order equation of motion with colored noise is modified to four coupled first-order differential equations. Based on the modified equation of motion, corresponding 4-D FPK equation results in. Secondly, the EPC method, originally with approximate solution containing two state variables for cases of white noise excitations, is developed to extend variables to four state variables with additional unknown coefficients for cases of colored noise excitations. As a result, much more sophisticated solution procedure and computer programmer are produced. On the other hand, MCS method is performed to verify the efficiency of the developed EPC method. Thirdly, two examples are considered. One example is about nonlinear oscillator under Gaussian colored noise, the other is about nonlinear oscillator under non-Gaussian colored noise. Stationary approximate PDF solutions with the developed EPC method, EQL method and MCS method are obtained and compared with each other. Moveover, statistical parameter mean-up crossing rate (MCR) is taken into account.

2. Problem formulation and developed EPC solution procedure

As white noise process has infinity energy, it cannot exist in practice. To demonstrate the real situations, colored noise excitation is taken into account. Consider the following oscillator

$$\ddot{X} + h_0(X,\dot{X}) + h_i(X)W_i(t) = C(t), \tag{3}$$

where $X, \dot{X}$ and $\ddot{X}$ represent displacement, velocity and acceleration, $h_0(X,\dot{X})$ and $h_i(X)$ are deterministic linear or nonlinear functions. $W_i(t)$ are independent Gaussian white noise excitations with PSDs S_{ii}, $C(t)$ is colored noise excitation with band-limited PSD function $S(\omega)$. For the sake of simplifying the analysis, white noise $W_i(t)$ and colored noise $C(t)$ are assumed to