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Research paper

Wada property in systems with delay

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ABSTRACT

Delay differential equations take into account the transmission time of the information. These delayed signals may turn a predictable system into chaotic, with the usual fractalization of the phase space. In this work, we study the connection between delay and unpredictability, in particular we focus on the Wada property in systems with delay. This topological property gives rise to dramatic changes in the final state for small changes in the history functions.

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1. Introduction

The objective of this work is twofold: on the one hand, we study the effects of the delay in nonlinear dynamical systems, focusing on their associated uncertainty and on the other hand, we study the emergence of the Wada property when delays are involved.

Delay differential equations (DDEs) take into account the time taken by systems to sense and react to the information they receive, in other words, the information transmission cannot be instantaneous. For practical purposes, these lags can often be ignored when their timescales are very small compared to the dynamics of the system. However, there are situations where large delays cannot be overlooked: genetic oscillators [1], neuron networks [2], respiratory and hematopoietic diseases [3], electronic circuits [4], optical devices [5], engineering applications [6], etc. Delay differential equations provide a very useful tool for the modelling of the previous examples [7,8]. Moreover, they are able to display such interesting kind of dynamics as deterministic Brownian motion [9], hyperchaos [10] and many cooperative effects [11–13].

A crucial feature of DDEs is that they need an infinite set of initial conditions to be integrated. This set is usually called the *history* and provides the state of the system before the action of the delayed terms. Sometimes the history is set randomly, although given the sensitivity of some systems with delay and the difficulties arising when an infinite set of initial random points is needed, the choice of random histories is a delicate issue. A better option is to set the history as the solution of the system without the delayed terms. Another possibility is to consider history functions described by some parameters which are properly chosen for each physical situation. For more details about history functions, see the Appendix.

A convenient way to handle this infinite number of initial conditions is to define the history functions characterized by a finite number of parameters. This supposes a huge difference with respect to nonlinear systems modelled with ordinary differential equations (ODEs), due to the strong difference between the space of the history functions and the real phase

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space. Therefore, basins of attraction, which are a very powerful tool to study sensitivity in dissipative systems, have a different nature in DDEs.

In dissipative dynamical systems defined by ODEs, the basins of attraction register in a plot the attractors reached by different initial conditions. In DDEs the same idea can be exploited: we can plot basins of attraction varying the parameters of the history functions. These basins are subspaces of the infinite dimensional space of history functions, but still can provide much information about the sensitivity of the system. Only a few authors have studied basins of attraction in DDEs [14–16]. In this work, we study how the delay can induce uncertainty in the system, and we pay special attention to the appearance of the Wada property in these basins of attraction made out of different history functions.

The Wada property is a topological property that can be stated as follows: given more than two open sets, they all share the same boundary. This situation is very counter-intuitive, since most boundaries are only between two sets. In some cases there might be some points or regions that separate more than two open sets, but the case where every point in the boundary is in the boundary of all the sets is unique.

This topological property was first reported by Kunizo Yoneyama [17], who attributed its discovery to his teacher Takeo Wada. Later on, the research work done by James A. Yorke and coworkers contributed to relate the Wada property with nonlinear dynamics [18–21]. They found that the basins of attraction of simple physical systems such as the forced pendulum can possess the Wada property [20]. In these seminal works, they gave a topological argument showing that the origin of this property was an unstable manifold crossing all the basins [19]. After that, the Wada property has been found in a variety of models associated to physical phenomena [22–26].

The interest in the Wada basin boundaries lies on the fact that they are the most entangled boundaries one can imagine, since they separate all the basins at the same time. Therefore, small perturbations near the boundaries can lead to any of the different attractors of the system. Recently, Zhang and collaborators [27–29] have studied an intermediate situation, where some regions have the Wada property but others do not, so they call this situation partial Wada basins.

The Wada property also appears in systems with more than two degrees of freedom [30–32]. In these cases the basins have more than two dimensions and the subspaces generally show the disconnected Wada property: the different basins share the same boundary but they are disconnected. These disconnected Wada basins can be analyzed by means of the techniques developed in [33].

The aim of this work is to investigate the connection between delay and unpredictability, looking also for the Wada property in systems with delay. We organize this research as follows. In Sections 2 and 3 we introduce two delayed systems that present different degrees of the Wada property. Finally, in Section 4 we briefly summarize and discuss our main results.

2. Forced delayed action oscillator

We start by studying an apparently simple system sometimes called the delayed action oscillator (DAO). It is a single variable system with a double-well potential and a linear delayed feedback with constant time delay τ, that we will denote $x(t-\tau)$. It can be stated as follows,

$$\dot{x} + x((1+\alpha)x^2 - 1) - \alpha x(t-\tau) = 0. \tag{1}$$

where $\alpha, \tau \in \mathbb{R}$. Boutle et al. [34] proposed this model in the context of the ENSO (El Niño Southern Oscillation) phenomenon, where the variable x represents the temperature anomaly of the ocean's surface. In [35] the authors analyze the stability and bifurcations of this dissipative system by a center manifold reduction. They demonstrate that as the delay increases beyond a critical value τ_c, the steady state solution $x = 0$ can undergo a Hopf bifurcation, turning the equilibria unstable and giving rise to a limit cycle. Without the delayed term, this system would be a one-dimensional ODE and could not oscillate, but the linear delayed feedback makes the system infinite dimensional allowing oscillatory dynamics. In the case that $\alpha > -1$ and $\tau > \tau_c$ the limit cycle coexists with two stable fixed points, so the system can be multistable. To visualize the situation in a plot, we choose the following family of history functions defined by two parameters A and B,

$$x(t) = A + Bt, \ \forall t \in [-\tau, 0]. \tag{2}$$

Unless specified, this linear equation will be the family of history functions chosen by default along the paper. Now, we can compute a basin of attraction varying A and B. Fig. 1(a) represents the basin for $\alpha = -0.95$ and $\tau = 1.065$, below the critical value τ_c. It is interesting to notice the analogy between the DAO and the well-known Duffing oscillator

$$\ddot{x} + \gamma\dot{x} + x(x^2 - 1) = 0. \tag{3}$$

The structure of its basin of attraction is very similar to the DAO model as shown in Fig. 1(b). However, it is important to notice that the two basins have a different nature: we have the real phase space for the Duffing oscillator and a slice of the infinite space of history functions in the case of the DAO.

At this point it is important to make a connection with Ref. [22]. In that work, Aguirre and Sanjuán studied the Duffing oscillator driven by a periodic forcing $F\sin\omega t$ on the right hand side of Eq. (3). They showed that for a carefully chosen set of parameters ($\gamma = 0.15, \omega = 1, F \in (0.24, 0.26)$) the system can display the Wada property. Making a naive analogy, it is plausible that we will encounter the same effect by including the periodic forcing in the DAO such as

$$\dot{x} + x((1+\alpha)x^2 - 1) - \alpha x(t-\tau) = F\sin\omega t. \tag{4}$$

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