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Research paper

Motion in an asymmetric double well

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ABSTRACT

The problem of the periodic motion of a particle in an asymmetric double-well (quartic) potential is solved explicitly in terms of the Weierstrass and Jacobi elliptic functions. While the solution of the orbital motion is expressed simply in terms of the Weierstrass elliptic function, the period of oscillation is more directly expressed in terms of periods of the Jacobi elliptic functions.

1. Introduction

The double-well potential is an important paradigm in physics and chemistry. In classical mechanics, for example, it is used to study the motion of a particle either trapped in one of the two wells or moving with energy above the height of the barrier that separates the two wells [1]. For particles trapped in either of the two wells, the symmetry of the double-well potential implies that the oscillation periods are equal. When quantum tunneling through the barrier of a symmetric double-well potential is taken into account, however, the coupling of the solutions of the Schrödinger equation for this two-state quantum problem [2–5] leads to a splitting of the degenerate energy level. The ammonia-inversion problem is a well-known example of a symmetric double-well potential [6,7]. Important quantum applications of a symmetric double-well potential include atom interferometry using Bose–Einstein condensates [8,9].

The addition of asymmetry in the double-well potential, where the two minima are no longer at the same energy level, introduces nontrivial modifications of the standard asymptotic treatment of quantum tunneling and energy-level splitting [7,10–12]. Quantum applications of an asymmetric double-well potential also provide a natural generalization of atom interferometry using Bose–Einstein condensates [13–15].

The purpose of the present paper is to explore the classical orbits of a particle moving in an asymmetric double-well potential represented by a quartic polynomial. In particular, on an energy level that allows periodic motion in the shallow and deep wells, we prove that the oscillation periods for these orbits are equal. This result was recently derived by Levi [16] based on the process of contour deformation on the Riemann sphere [17]. Here, we prove this result by deriving an explicit solution of the asymmetric double-well problem expressed in terms of the Weierstrass elliptic functions [18–21].

The remainder of this paper is organized as follows. In Section 2, we characterize an asymmetric quartic potential in terms of its two minima ($V_c \leq V_a$) and its single maximum $V_b \geq V_a$. Here, the asymmetric quartic potential is parameterized by a single asymmetry parameter δ that vanishes in the symmetric case. In Section 3, the four turning points (ξ_1, ξ_2, ξ_3,

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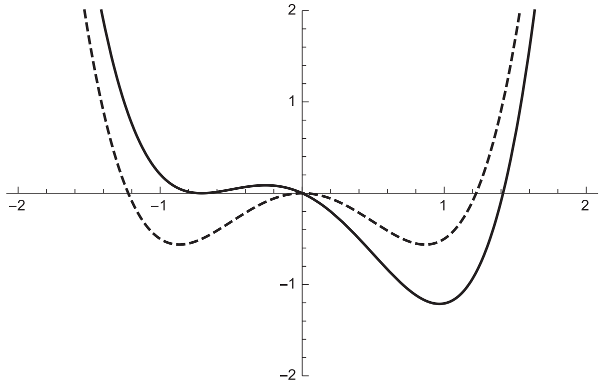

Fig. 1. Double-well potential (1) for fixed values of $\delta \geq 0$: symmetric potential (dashed line) for $\delta(\varphi = \pi/2) = 0$ and asymmetric potential (solid line) for $\delta(\varphi = \pi/4) = 1/\sqrt{2}$.

ξ_4) for the asymmetric quartic potential are expressed in terms of simple formulas parameterized by the energy value E and the asymmetry parameter δ. In Section 4, solutions of the motion in the asymmetric quartic potential are given in terms of the Weierstrass elliptic function by transforming the quartic energy equation into the standard Weierstrass cubic equation [19,20]. In Section 5, we express the solutions for the asymmetric quartic orbits in terms of the Jacobi elliptic functions [22] with periods expressed in terms of the complete elliptic integral of the first kind. Lastly, in Section 6, we briefly discuss a classical application of the present work when the asymmetry in the double-well potential is time dependent (e.g., by considering the undamped Duffing oscillator driven to chaotic behavior by a periodic force [23–25]).

2. Local minima and maxima of an asymmetric double well

We consider an asymmetric double-well potential

$$V(x) \equiv x^4 - \frac{3}{2}x^2 - \delta\, x, \tag{1}$$

where the parameter δ represents the asymmetry in the potential. Fig. 1 shows that if δ is in the range $|\delta| \leq 1$, the potential (1) has two local minima ($x_c > x_a$) and one local maximum x_b (with $x_a < x_b < x_c$), which are solutions of the cubic equation

$$V'(x_i) = 4\,x_i^3 - 3\,x_i - \delta = 0. \tag{2}$$

The case $|\delta| > 1$, when a single local minimum remains, will not be considered in this work. The cubic Eq. (2) is a special case of the generic Weierstrass cubic equation $4\,x^3 - g_2\,x - g_3 = 0$ discussed in Appendix A.

It is convenient to parameterize the asymmetry parameter in terms of the phase $0 \leq \varphi \leq \pi$:

$$\delta(\varphi) \equiv \cos\varphi = 4\cos^3\left(\frac{\varphi}{3}\right) - 3\cos\left(\frac{\varphi}{3}\right), \tag{3}$$

so that the cubic roots of Eq. (2) are thus expressed as

$$\left.\begin{array}{l} x_a(\varphi) = -\cos[(\pi - \varphi)/3] \\ x_b(\varphi) = -\cos[(\pi + \varphi)/3] \\ x_c(\varphi) = \cos(\varphi/3) \end{array}\right\}, \tag{4}$$

and the roots (4) satisfy the condition $x_a + x_b + x_c = 0$.

The potential minima (at $x_a < x_c$) and maximum (at $x_a < x_b < x_c$) are

$$V_i(\varphi) = -\frac{3}{4}x_i(\varphi)\,(x_i(\varphi) + \cos\varphi) \equiv \frac{9}{16}\epsilon_i(\varphi), \tag{5}$$

where we used $x_i^3 = (3\,x_i + \delta)/4$ obtained from Eq. (2). Fig. 2 shows the critical (normalized) energy levels

$$-\frac{8}{3} \leq \epsilon_c(\varphi) \leq \epsilon_a(\varphi) \leq \epsilon_b(\varphi) \leq \frac{1}{3} \tag{6}$$

as functions of the asymmetry parameter $\delta(\varphi) = \cos\varphi$ from $\delta = 0$ to $\delta = 1$.