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Research paper

Amplitude death, oscillation death, wave, and multistability in identical Stuart–Landau oscillators with conjugate coupling

Wenchen Han, Hongyan Cheng, Qionglin Dai, Haihong Li, Ping Ju, Junzhong Yang*

School of Science, Beijing University of Posts and Telecommunications, Beijing 100876, People's Republic of China

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ABSTRACT

In this work, we investigate the dynamics in a ring of identical Stuart–Landau oscillators with conjugate coupling systematically. We analyze the stability of the amplitude death and find the stability independent of the number of oscillators. When the amplitude death state is unstable, a large number of states such as homogeneous oscillation death, heterogeneous oscillation death, homogeneous oscillation, and wave propagations are found and they may coexist. We also find that all of these states are related to the unstable spatial modes to the amplitude death state.

1. Introduction

Pattern formation in nonequilibrium systems is always a hot subject in diverse extended systems. Generally, patterns can arise from the instability of an equilibrium state [1]. Depending on the instability, the patterns may be stationary or oscillatory in time and homogeneous or structured in space. Coupled nonlinear oscillators may provide a platform for the investigation on pattern formation.

Coupled nonlinear oscillators also provide a platform for investigating collective behaviors in physical, chemical and biological systems [2–4] such as oscillation quenching, synchronization, and others. There are two types of oscillation quenching, amplitude death (AD) and oscillation death (OD). Generally, AD is a complete cessation of oscillations, which refers to a situation where coupled oscillators cease to oscillate and go to an equilibrium solution, the origin of the system [6,7]. TAD state may be induced by several different ways such as a large mismatch of the oscillator frequencies [7,8], or the existence of time delay in the coupling [9–12], and the coupling of identical oscillators through dissimilar (or conjugate) variables [13–15]. On the other hand, OD, which has been actively studied recently [5,6,16], is thought to have a significantly different background of the occurrence compared to AD. OD phenomenon is believed as a result of the system's symmetry breaking and is manifested as a stabilized inhomogeneous steady state [16].

In coupled nonlinear oscillators, oscillators always interact with each other through the same variables. However, coupling via dissimilar variables is also natural in real situations [17,18] and this type of coupling is called conjugate coupling. Kim and Roy conducted coupled-semiconductor-laser experiments [19], where the photon intensity fluctuation from one laser was used to modulate the injection current of the other, and vice versa. Singla et al. designed an setup consisting of

* Corresponding author. Tel.: +86 15210830535.
E-mail addresses: hwcurey@gmail.com, ulyssesurey@163.com (W. Han), jzyang@bupt.edu.cn (J. Yang).

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two Chua circuits with conjugate coupling [20] and explored the dynamics both experimentally and theoretically. Additionally, the conjugate coupled Chua circuits can be generalized to consist of a large number of units in a straightforward way. The investigation on the dynamics in conjugate coupled oscillators concerned with amplitude death, synchronization, and so on have been done in some recent literatures [21–24].

In this work, we consider the AD state in N identical Stuart–Landau oscillators with the conjugate coupling and investigate the pattern formation when the AD state becomes unstable. The model with $N = 2$ has been employed by Karnatak et al. to demonstrate that the AD state can be realized in coupled identical oscillators [13]. Here, we consider arbitrary N. We systematically analyze the stability of the AD state and find that the stability of the AD state is independent of N. Large numbers of states such as homogeneous OD state, heterogeneous OD states, homogeneous oscillation, and wave propagations are found when the AD state is unstable. Interestingly, different states may coexist. Furthermore, we find that these states originate from different the unstable spatial modes of the AD state.

2. Analysis and discussion

We consider N identical Stuart–Landau oscillators sitting on a ring. Each oscillator interacts with its nearest neighbors via a diffusive conjugate coupling. The motion equation of the system is described as

$$\begin{aligned} \dot{x}_i &= (1 - x_i^2 - y_i^2)x_i - \omega y_i + \epsilon(y_{i+1} + y_{i-1} - 2x_i) \\ \dot{y}_i &= (1 - x_i^2 - y_i^2)y_i + \omega x_i + \epsilon(x_{i+1} + x_{i-1} - 2y_i) \end{aligned} \tag{1}$$

where $i = 1, \ldots, N$. The periodic boundary condition is imposed on the system. ϵ is the coupling strength. When $\epsilon = 0$, each oscillator oscillates at the frequency ω and the origin $x = 0$ and $y = 0$ is an unstable equilibrium.

2.1. Amplitude death

To investigate the dynamics in the model (1), we begin with the AD state. For convenience, we let $\mathbf{s}_i = (x_i, y_i)$ and $\mathbf{s}_i = \mathbf{0}$ for all i in the AD state. We first consider the stability of the AD state. To do it, we perturb the AD state by letting $\mathbf{s}_i = \xi_i$. Then the evolution of $\boldsymbol{\xi}_i$ follows

$$\frac{d}{dt}\xi_i = [\mathcal{D}\mathbf{F}(\mathbf{0}) - 2\epsilon\mathcal{D}_1]\xi_i + \epsilon\mathcal{D}_2\sum_j \mathcal{C}_{ij}\xi_j \tag{2}$$

where $\mathcal{D}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\mathcal{D}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. $\mathcal{C}$ is an $N \times N$ coupling matrix with zero elements except that $\mathcal{C}_{i,i+1} = \mathcal{C}_{i-1,i} = 1$ describing the interaction among oscillators. $\mathcal{D}\mathbf{F}(\mathbf{0}) = \begin{pmatrix} 1 & -\omega \\ \omega & 1 \end{pmatrix}$ is the Jacobian matrix of a Stuart–Landau oscillator on $\mathbf{s} = 0$.

To analyze the stability of the AD state, we adopt the methods in Refs. [25,26]. The coupling matrix $\mathcal{C}$ can be diagonalized to $\mathcal{C}\phi_\alpha = \lambda_\alpha\phi_\alpha$ $(\alpha = 0, 1, \ldots, N-1)$. For the ring structure, the coupling matrix $\mathcal{C}$ has its eigenvalues $\lambda_\alpha = 2\cos\frac{2\alpha\pi}{N}$ and eigenvectors $\phi_\alpha = (1, \phi_\alpha^1, \ldots, \phi_\alpha^{N-1})$ representing spatial modes with different wavenumbers. Nonuniform perturbation $\boldsymbol{\xi}_i$ can be expanded onto the eigenvectors ϕ_α of $\mathcal{C}$ and we have $\xi_i = \Sigma_{\alpha=0}^{N-1}\eta_\alpha\phi_\alpha^i$. Here $\boldsymbol{\eta}_\alpha$ are time-dependent coefficients. Substituting the expansion into Eq. (2) and equating the coefficients for each ϕ_α, we have N independent linear equations for different spatial modes

$$\frac{d}{dt}\eta_\alpha = [(\mathcal{D}\mathbf{F}(\mathbf{0}) - 2\epsilon\mathcal{D}_1) + \epsilon\lambda_\alpha\mathcal{D}_2]\eta_\alpha = \mathcal{D}\mathbf{F}^\alpha(\mathbf{0})\eta_\alpha. \tag{3}$$

The matrix $\mathcal{D}\mathbf{F}^\alpha(\mathbf{0})$ depends not only on the parameters of local dynamics but also on the spatial mode. The largest eigenvalue of $\Lambda^{(\alpha)}$ of the matrix $\mathcal{D}\mathbf{F}^\alpha(\mathbf{0})$ gives the growth rate of the αth spatial mode. When all $Re(\Lambda^{(\alpha)})$ are negative, the AD state is stable. Otherwise, the AD state lose its stability. For spatial modes with positive $Re(\Lambda^{(\alpha)})$, we call them unstable modes to the AD state. For an unstable mode with zero $Im(\Lambda^{(\alpha)})$, the perturbation possibly gives rise to a stationary spatial pattern, while a oscillatory pattern is possible for an unstable mode with nonzero $Im(\Lambda^{(\alpha)})$. Furthermore, if there is only one unstable mode to the AD state, the final pattern always has the same spatial structure as the unstable mode no matter whether the pattern is stationary or oscillatory. However, if there are more than one unstable modes, the final pattern is determined by the competition among these modes. The mode with the highest $Re(\Lambda^{(\alpha)})$ tends to win the competition.

For the model (1), the matrix $\mathcal{D}\mathbf{F}^\alpha(\mathbf{0})$ takes the form $\mathcal{D}\mathbf{F}^\alpha(\mathbf{0}) = \begin{pmatrix} 1-2\epsilon & -\omega+\epsilon\lambda_\alpha \\ \omega+\epsilon\lambda_\alpha & 1-2\epsilon \end{pmatrix}$ and, by introducing the notation $Tr(\mathcal{D}\mathbf{F}^\alpha(\mathbf{0}))$ and $Det(\mathcal{D}\mathbf{F}^\alpha(\mathbf{0}))$ for the trace and the determinant of the matrix $\mathcal{D}\mathbf{F}^\alpha(\mathbf{0})$, we have $\Lambda_\alpha = [Tr(\mathcal{D}\mathbf{F}^\alpha(\mathbf{0})) \pm \sqrt{Tr(\mathcal{D}\mathbf{F}^\alpha(\mathbf{0}))^2 - 4Det(\mathcal{D}\mathbf{F}^\alpha(\mathbf{0}))}]/2$. When $\epsilon < |\omega/\lambda_\alpha|$, the AD state is a focus to the spatial mode ϕ_α. Negative $Re(\Lambda^{(\alpha)})$ requires $Tr(\mathcal{D}\mathbf{F}^\alpha(\mathbf{0}))$ to be negative and $Det(\mathcal{D}\mathbf{F}^\alpha(\mathbf{0}))$ to be positive. The former gives $\epsilon > \epsilon_{c,T} = 1/2$ and the latter gives $\lambda_\alpha^2 < [\omega^2 + (2\epsilon - 1)^2]/\epsilon^2$. Since $\lambda_\alpha = 2\cos(2\alpha\pi/N)$, the second inequality is first violated by λ_0 for odd N or by λ_0 and $\lambda_{N/2}$ for even N at $\epsilon_{c,D} = (\omega^2 + 1)/4$. For larger ϵ, the second inequality may be violated by more and more spatial modes. According to the stability condition of the AD state requiring all spatial modes to be stable, the AD state is stable for the coupling strength in the range of $\epsilon \in (\epsilon_{c,\,T}, \epsilon_{c,\,D})$ when $\omega > 1$. When $\omega < 1$, $\epsilon_{c,\,T} > \epsilon_{c,\,D}$ and the AD state is unstable no matter what ϵ is. The stability regime in the parameter space ω and ϵ is presented in Fig. 1. To be noted, the stability condition for the stable AD state has been derived in the work [13] for $N = 2$. Here, we have shown that the stability of the AD state is independent of N in conjugate coupled Stuart–Landau oscillators.