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Model of the Newtonian cosmology: Symmetries, invariant and partially invariant solutions

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ABSTRACT

Symmetry group of the equation system of ideal nonrelativistic self-gravitating fluid with zero pressure is calculated. Submodel invariant under the subgroup of rotations SO(3) is built and symmetry group of the factorsystem is calculated. A particular analytical invariant solution of the factorsystem is obtained.

1. Introduction

Model of the Newtonian cosmology is basic in the study of large-scale structure of the Universe [1–7]. The model is a system of equations of ideal nonrelativistic self-gravitating fluid with zero pressure, density ρ, velocity $\vec{v}$ and gravitational potential Φ [2]

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \nabla.(\rho \vec{v}) = 0, \\ \dfrac{\partial \vec{v}}{\partial t} + (\vec{v}.\nabla)\vec{v} + \nabla(\Phi) = 0, \\ \Delta \Phi = 4\pi \gamma \rho. \end{cases} \tag{1}$$

The first equation is the equation of continuity, the second – Euler equation, the third – Poisson equation, Δ – Laplace operator, γ is the gravitational constant. The present model is nonrelativistic, but adequately describes the evolution of inhomogeneities in the density of matter in the Universe, because the peculiar velocity and gravitational potential, as is known, do not reach relativistic values (for example, see a detailed discussion in the review [1]. At the present time various approximations of the solution of system (1) are well studied, the best known of them is Zeldovich "pancakes" model [1,3]. The aim of our work is systematically study the system of equations (1) by methods of group analysis that will yield new exact analytical solutions that not only going beyond perturbation theory but also useful for testing numerical methods.

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Due to the symmetry in the model of Newtonian cosmology one can apply the SUBMODELS program, similar to Ovsyannikov's program in gas dynamics. So, the system (1) will present as a "big model" [10]. The program SUBMODELS allows in principle to obtain all possible invariant and partially invariant solutions of the "big" model and to analyze their physical meaning. This, in our opinion, is the advantage of the methodology of group analysis compared with other methods of mathematical modeling.

2. Lie point symmetries

Rewrite the system (1) in Cartesian coordinates in dimensionless variables:

$$\begin{cases} \rho_t + \rho(u_x + v_y + \omega_z) + u\rho_x + v\rho_y + \omega\rho_z = 0, \\ u_t + uu_x + vu_y + \omega u_z + \Phi_x = 0, \\ v_t + uv_x + vv_y + \omega v_z + \Phi_y = 0, \\ \omega_t + u\omega_x + v\omega_y + \omega\omega_z + \Phi_z = 0, \\ \Phi_{xx} + \Phi_{yy} + \Phi_{zz} = \rho, \end{cases} \tag{2}$$

where x, y, z – Cartesian coordinates, t – time, u, v, ω – the velocity components.

Generator of the group will be sought in the form

$$\hat{X} = \xi^{(x)}\partial_x + \xi^{(y)}\partial_y + \xi^{(z)}\partial_z + \xi^{(t)}\partial_t + \eta^{(\rho)}\partial_\rho + \eta^{(\Phi)}\partial_\Phi + \eta^{(u)}\partial_u + \eta^{(v)}\partial_v + \eta^{(\omega)}\partial_\omega,$$

where ξ and η are components of the tangent vector field, ∂ – operator of differentiation in the corresponding variable. Calculation according to standard algorithm Lie–Ovsyannikov [8] with help of package GeM [9] shows that the system allows for an infinite-dimensional algebra with the components of the tangent vector field

$$\begin{aligned} \xi^{(x)} &= F_1(t) + (C_1 - C_4)x + C_2 y + C_{10}, \\ \xi^{(y)} &= F_2(t) + (C_1 - C_4)y - C_2 x + C_6 z + C_7, \\ \xi^{(z)} &= F_3(t) + (C_1 - C_4) + C_8 t - C_3 x - C_6 y + C_9, \\ \xi^{(t)} &= -C_4 t + C_5, \\ \eta^{(\rho)} &= 2C_4\rho, \\ \eta^{(\Phi)} &= 2C_1\Phi - F_{1tt}x - F_{2tt}y + F_4(t) + F_{3tt}z, \\ \eta^{(u)} &= C_1 u + C_2 v + C_3\omega + F_{1t}, \\ \eta^{(v)} &= C_1 v - C_2 u + C_6\omega + F_{2t}, \\ \eta^{(\omega)} &= C_1\omega - C_3 u - C_6 v + F_{3t} + C_8, \end{aligned}$$

where $F_i(t)$ – arbitrary functions, C_k – arbitrary constants. This infinite-dimensional algebra contains a 13-dimensional subalgebra, say, L_{13} with generator translations

$$\hat{X}_1 = \partial_x, \quad \hat{X}_2 = \partial_y, \quad \hat{X}_3 = \partial_z, \quad \hat{X}_4 = \partial_t, \quad \hat{X}_5 = \partial_\Phi;$$

Galilean transformations

$$\hat{X}_6 = t\partial_x + \partial_u, \quad \hat{X}_7 = t\partial_y + \partial_v, \quad \hat{X}_8 = t\partial_z + \partial_\omega;$$

rotations

$$\begin{aligned} \hat{X}_9 &= y\partial_z - z\partial_y + v\partial_\omega - \omega\partial_v, \quad \hat{X}_{10} = z\partial_x - x\partial_z + \omega\partial_u - u\partial_\omega, \\ \hat{X}_{11} &= x\partial_y - y\partial_x + u\partial_v - v\partial_u; \end{aligned}$$

and dilatations

$$\begin{aligned} \hat{X}_{12} &= 2\Phi\partial_\Phi + u\partial_u + v\partial_v + \omega\partial_\omega + x\partial_x + y\partial_y + z\partial_z, \\ \hat{X}_{13} &= -2\rho\partial_\rho + x\partial_x + y\partial_y + z\partial_z + t\partial_t. \end{aligned}$$

The corresponding commutator table for a Lie Algebra L_{13} has the form Table 1.

To realize the SUBMODEL program [10] it is necessary to calculate the optimal system of subalgebras for the 13-dimensional algebra $\hat{X}_1, \ldots, \hat{X}_{13}$, which is the subject of a separate study. In this paper, we focus on the submodels, which is invariant under the rotation group SO(3) with generators $(\hat{X}_9, \hat{X}_{10}, \hat{X}_{11})$.

3. Invariant submodel SO(3)

Invariants of the group SO(3) are

$$\Phi, \rho, t, r = \sqrt{x^2 + y^2 + z^2}, \quad |\vec{v}| = \sqrt{u^2 + v^2 + \omega^2} \equiv U, \quad s = \vec{r} \cdot \vec{v}.$$