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Exponentials and Laplace transforms on nonuniform time scales

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ABSTRACT

We formulate a coherent approach to signals and systems theory on time scales. The two derivatives from the time-scale calculus are used, i.e., nabla (forward) and delta (backward), and the corresponding eigenfunctions, the so-called nabla and delta exponentials, computed. With these exponentials, two generalised discrete-time Laplace transforms are deduced and their properties studied. These transforms are compatible with the standard Laplace and Z transforms. They are used to study discrete-time linear systems defined by difference equations. These equations mimic the usual continuous-time equations that are uniformly approximated when the sampling interval becomes small. Impulse response and transfer function notions are introduced. This implies a unified mathematical framework that allows us to approximate the classic continuous-time case when the sampling rate is high or to obtain the standard discrete-time case, based on difference equations, when the time grid becomes uniform.

1. Introduction

The analysis of nonuniformly sampled data is a very important task having large spread application in fields like astronomy, seismology, paleoclimatology, genetics and laser Doppler velocimetry [5]. The “jitter” in Telecommunications is a well-known problem [24]. Very interesting is the heart rate variability of the signal obtained from the “R” points [37]. Traditionally, most interesting techniques for dealing with this kind of signals pass by interpolation, to obtain a continuous-time signal that is analysed by current methods [2,22,23]. An alternative approach proposed in [25] allows a conversion from irregular to regular samples, maintaining the discrete-time character. Other approaches, include the study of difference equations with fractional delays [6,32]. However, no specific tools for dealing directly with such signals were developed. In particular, no equivalent to the Laplace or Z transforms were proposed. As well-known, the use of Laplace and Z transforms, to solve differential and difference linear equations, is very common in almost all scientific activities [38]. Normally, uniform time scales are used, but frequent applications use nonuniform scales. This makes important to obtain generalisations of such transforms for other kinds of scales. Some attempts have been made [1,3,4,11,12], but let us an unsatisfactory feeling: they are not true generalisations of the classic formulations. The main difficulty is in the starting point, i.e., the exponentials used to define the transforms. Usually, on time scales, causal exponentials are used instead of two-sided exponentials

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[18,26,27,31]. On the other hand, no correct interplay between nabla and delta derivatives and exponentials and transforms has been established. Such interplay was stated for fractional derivatives in the recent paper [36].

Here, in a first step, we clarify nabla and delta definitions and their meaning and relation with causality. With nabla (causal) and delta (anti-causal) derivatives, we define corresponding linear systems. Each concept is used to define two exponentials over the whole time scale and not only above or below a given time reference. With each exponential, a given transform is defined. We start from the inverse transform and only later we define the direct transform. With the nabla exponential, we define the inverse nabla transform through a Mellin-like integral on the complex plane. The direct nabla transform is defined with the help of the delta exponential. For the delta transform we reverse the exponentials. Having defined the exponentials, we study the question of existence, arriving to the concept of region of convergence. The unicity of the transforms is also investigated. This lead us to generalise the convolution and correlation concepts with the help of equivalent time scales. The concept of transfer function, as the eigenvalue corresponding to the respective exponential, is introduced. The inverse Laplace transform is the so called impulse response, i.e., the response of the system when the input is a delta function. We consider also the conversion from one time scale to another one that is equivalent to it. In passing, we prove the existence of no periodicity and, consequently, the inability to define Fourier transforms and series. This is the main drawback of the theory.

2. On the calculus on time scales

A powerful approach into the continuous/discrete unification/generalisation was introduced by Aulbach and Hilger through the calculus on *measure chains* [3,28]. However, the main popularity was gained by the calculus on *time scales* [4,14,16]. These are nonempty closed subsets $\mathbb{T}$ of the set $\mathbb{R}$ of real numbers, particular cases of measure chains. We remark that the name may be misleading, since the term *scale* is used in Signal Processing with a different meaning. On the other hand, in many problems we are not dealing with time.

Let t be the current instant. Using the language of the time-scale calculus, the previous next instant is denoted by $\rho(t)$. Similarly, the next following point on the time scale $\mathbb{T}$ is denoted by $\sigma(t)$. One has

$$\rho(t) = t - \nu(t), \quad \sigma(t) = t + \mu(t),$$

where $\nu(t)$ and $\mu(t)$ are called the graininess functions. We avoid here the use of the terms forward and backward graininess, since they have meanings that are different from the ones used in Signal Processing applications. Let us define

$$\nu^0(t) := 0, \quad \nu^n(t) := \nu^{n-1}(t) + \nu(t - \nu^{n-1}(t)), \quad n \in \mathbb{N},$$

and $\rho^0(t) := t$, $\rho^n(t) := \rho\big(\rho^{n-1}(t)\big)$, $n \in \mathbb{N}$. Note that $\nu^1(t) = \nu(t)$ and $\rho^1(t) = \rho(t)$. When moving into the past, we have

$$\begin{aligned}
\rho^0(t) &= t = t - \nu^0(t),\\
\rho^1(t) &= \rho(t) = t - \nu(t) = t - \nu^1(t),\\
\rho^2(t) &= \rho(\rho(t)) = \rho(t) - \nu(\rho(t)) = t - \nu(t) - \nu(t - \nu(t)) = t - \nu^2(t),\\
&\ \ \vdots\\
\rho^n(t) &= t - \nu^n(t).
\end{aligned}$$

Moving into the future, the definitions and results are similar:

$$\begin{aligned}
&\mu^0(t) := 0, \quad \mu^n(t) := \mu^{n-1}(t) + \mu(t + \mu^{n-1}(t)),\\
&\sigma^0(t) := t, \quad \sigma^n(t) := \sigma(\sigma^{n-1}(t)),
\end{aligned}$$

$n \in \mathbb{N}$, and we have $\sigma^n(t) = t + \mu^n(t)$.

Example 1. Let $\mathbb{T} = h\mathbb{Z}$, $h > 0$. In this case one has $\mu^n(t) = \nu^n(t) = nh$, $\sigma^n(t) = t + nh$, and $\rho^n(t) = t - nh$, $n = 0, 1, 2, \ldots$

With our notations, a time scale of isolated points is written in the form

$$\mathbb{T} = \{\ldots, \rho^m(t), \ldots, \rho^2(t), \rho^1(t), t, \sigma^1(t), \sigma^2(t), \ldots \sigma^n(t), \ldots\}.$$

This means that we can refer to all instants in the time scale $\mathbb{T}$ with respect to only one point t, taken as the reference. For example, if $\mathbb{T} = \mathbb{Z}$, such instant is usually taken as $t = 0$. We must be careful with the meaning of differences like $t - \tau$, because they can make no sense. Assume that t and τ are in $\mathbb{T}$ with $t > \tau$. This means that there exits an integer N such that $\tau = \rho^N(t) = t - \nu^N(t)$ and $t = \sigma^N(\tau) = \tau + \mu^N(\tau)$. It follows that $t - \tau = \nu^N(t) = \mu^N(\tau)$. Let us now define

$$\nu_n(t) := \rho^{n-1}(t) - \rho^n(t), \quad \mu_n(t) := \sigma^n(t) - \sigma^{n-1}(t),$$

as the nth values of the graininess functions from t. One can then see that the difference $t - \tau$ above is equal to the sum of the graininess values μ_n to go from τ to t,

$$t - \tau = \mu^N(\tau) = \mu(\tau) + \mu(\tau + \mu(\tau)) + \cdots = \sum_{n=1}^{N} \mu_n(\tau),$$

or, which is the same, the sum of the graininess values ν_n to go from t to τ,