Commun Nonlinear Sci Numer Simulat 39 (2016) 520–528

Contents lists available at ScienceDirect

Commun Nonlinear Sci Numer Simulat

journal homepage: www.elsevier.com/locate/cnsns

Research paper

Defining universality classes for three different local bifurcations

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A R T I C L E I N F O

Article history:
Received 21 July 2015
Revised 31 January 2016
Accepted 8 April 2016
Available online 25 April 2016

Keywords:
Scaling law
Critical exponents
Local bifurcations

A B S T R A C T

The convergence to the fixed point at a bifurcation and near it is characterized via scaling formalism for three different types of local bifurcations of fixed points in differential equations, namely: (i) saddle-node; (ii) transcritical; and (iii) supercritical pitchfork. At the bifurcation, the convergence is described by a homogeneous function with three critical exponents α, β and z. A scaling law is derived hence relating the three exponents. Near the bifurcation the evolution towards the fixed point is given by an exponential function whose relaxation time is marked by a power law of the distance of the bifurcation point with an exponent δ. The four exponents α, β, z and δ can be used to defined classes of universality for the local bifurcations of fixed points in differential equations.

1. Introduction

In nature there are many dynamical systems whose dynamics is controlled by a control parameter(s). As soon it(they) is(are) varied, the behavior of the asymptotic state can be changed in such a way that the qualitative structure of the solution is different after and before the change. The point to where such a qualitative change happens is defined as a bifurcation point.

The bifurcations can be classified in two main types [1,2]: (i) local and (ii) global bifurcations. A local bifurcation happens when a variation of the control parameter leads to a change of stability of a fixed point. Hence the topological modifications in the system can be confirmed by a local investigation near the fixed point. On the other hand, the global bifurcations are observed mainly when invariant structures collide with each other, like invariant manifold and chaotic attractor, leading to a destruction of the chaotic attractor. Such a destruction causes a major change in the global topology of the system which cannot be foreseen by a local analysis of fixed point. Examples of such global bifurcations include the so called crisis event [3–6]. In this paper we concentrate in the local bifurcations. Applications and observations of such bifurcations are wide in the literature and include laser [7–9], population dynamics [10,11], chemical reactions [12,13], electric circuits [14,15], discrete mappings [16,17] and many others [18–22].

Our main goal in this paper is to describe the behavior of the convergence to the fixed point and near it for three different local bifurcations, namely: (i) saddle-node; (ii) transcritical; and (iii) supercritical pitchfork. To do so, we consider two different approaches. The first one is phenomenological and shows that, at the bifurcation point, the convergence to the

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http://dx.doi.org/10.1016/j.cnsns.2016.04.008
1007-5704/

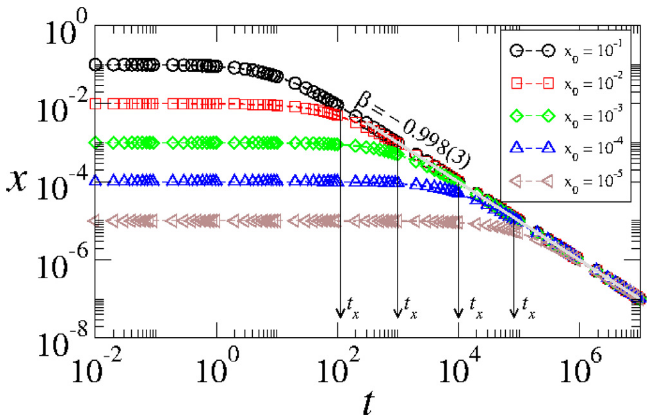

Fig. 1. Plot of x *vs.* t for Eq. (1) considering different values of x_0, as show in the figure.

fixed point is described by a homogeneous function with three critical exponents α, β and z. The exponent α characterizes the dynamics for short time. We shall show that in such a domain, the dynamics is strongly dependent on the initial distance from the fixed point. When the initial condition is given very close to the fixed point, the dynamics stays sticky in a constant plateau for a long time. Hence, the critical exponent α characterizes such a dynamics. For long time, the dynamics changes from this constant regime and enters in a decay towards the equilibrium. The speed of the convergence is given by the exponent β. The time marking the crossover from one region to the other is characterized by the exponent z. A scaling law is obtained relating all the three exponents in a single relation. Therefore the knowledge of only two of them is enough to determine the third. Near the bifurcation, the decay to the fixed point is exponential and the relaxation time is given in terms of a power law from the distance (in terms of the parameter) of the bifurcation with an exponent δ. The knowledge of the four critical exponents define to which universality class the bifurcations belongs to. The present formalism can be used as an alternative to classify the bifurcations mainly, but not only, in experimental systems when the modeling dynamical equations are not available or unknown.

The investigation of such a convergence is made by using two approaches. The first one is phenomenological where a set of three scaling hypotheses are presented together with a scaling function. Numerical simulations are made to obtain an estimation for the critical exponents. The second one comes from the exact solution of the differential equation, where the critical exponents emerge naturally from the dynamics. The second approach validates the first one. As discussed in Refs. [1,2], a generic vector field obtained by the solution of an equation of the type $\dot{x} = f(x, \mu)$, where x is a dynamical variable and μ is a control parameter and f is a nonlinear function, can be Taylor expanded near the fixed point. As a result, the dynamics can be matched in one of the three main differential equations: (a) $\dot{x} = \mu - x^2$ (saddle-node); (b) $\dot{x} = \mu x - x^2$ (transcritical); and (c) $\dot{x} = \mu x - x^3$ (supercritical pitchfork). This is possible due to a so called normal form theory [1,2].

The paper is organized as follows. In Section 2 we describe the general aspects of the procedure illustrating how the phenomenological investigation is made. The scaling law is derived in Section 2 too. Section 3 is reserved to described the critical exponents in the saddle-node bifurcation while the properties for the transcritical bifurcation are described in Section 4. The critical exponents for the supercritical pitchfork bifurcation is presented in Section 5. Our discussions and final remarks are given in Section 6.

2. Convergence to the fixed point

To start with, let us discuss the convergence to the fixed point for the following differential equation

$$\dot{x} = \mu - x^2. \tag{1}$$

There is a saddle-node bifurcation at $\mu = 0$. The fixed points are $x^*_{1,2} = \pm\sqrt{\mu}$ for $\mu \geq 0$. The fixed point x^*_1 is asymptotically stable for $\mu > 0$ while x^*_2 is unstable. It means that any neighboring initial condition of x^*_1 converge to it as time goes. On the other hand, the initial conditions chosen near x^*_2 apart from it as time evolves. The two questions we can pose are: (i) How fast is such a convergence? (ii) Could the convergence depends on the initial x_0 (the initial distance from the fixed point)? We show that the convergence does indeed depend on the initial x_0 and that the speed of convergence depends on the equation. Fig. 1 shows a typical behavior of the convergence to the fixed point (x *vs.* t) for Eq. (1), at the bifurcation in $\mu = 0$, considering different values for x_0, as shown in the figure.

We can see that, depending on the value of x_0, the orbit keeps sticky in a constant plateau for a long time. It then experiences a changeover and enters in a regime of decay given by a power law, converging asymptotically to the fixed point. The characteristic time t_x that marks the changeover from the constant plateau to the regime of decay depends also on the initial condition x_0. Let us then describe the convergence to the fixed point in a phenomenological way. Later on the paper we give a mathematical description for it.