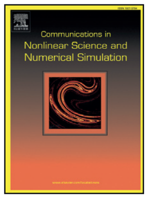
Commun Nonlinear Sci Numer Simulat 37 (2016) 31–61

Contents lists available at ScienceDirect

Commun Nonlinear Sci Numer Simulat

journal homepage: www.elsevier.com/locate/cnsns



Exact discretization by Fourier transforms

Vasily E. Tarasov*

Skobeltsyn Institute of Nuclear Physics, Lomonosov Moscow State University, Moscow 119991, Russia

ARTICLE INFO

Article history:
Received 21 July 2015
Revised 6 January 2016
Accepted 7 January 2016
Available online 15 January 2016

Keywords:
Discretization
Exact discretization
Finite differences
Infinite series
Fourier transform
Non-standard differences
Fractional derivative
Fractional integral
Fractional difference

ABSTRACT

A discretization of differential and integral operators of integer and non-integer orders is suggested. New type of differences, which are represented by infinite series, is proposed. A characteristic feature of the suggested differences is an implementation of the same algebraic properties that have the operator of differentiation (property of algebraic correspondence). Therefore the suggested differences are considered as an exact discretization of derivatives. These differences have a property of universality, which means that these operators do not depend on the form of differential equations and the parameters of these equations. The suggested differences operators allows us to have difference equations whose solutions are equal to the solutions of corresponding differential equations. The exact discretization of the derivatives of integer orders is given by the suggested differences of the same integer orders. Similarly, the exact discretization of the Riesz derivatives and integrals of integer and non-integer order is given by the proposed fractional differences of the same order.

1. Introduction

Differential and integral equations of integer and non-integer orders are very important for the theory of systems and processes in continua, media and fields. Equations with derivatives and integrals of non-integer orders [1–8] are actively used to describe processes and systems with power-law nonlocality and memory [12–18]. As it was shown in [14,36–39], the differential equations with derivatives of non-integer orders are directly connected with discrete models of systems with long-range interactions of power-law type. Interconnection between the equations of these discrete systems (lattices) and the fractional differential equations is proved by special transform operator that includes the Fourier series and integral transforms [36–39]. This approach has been applied to discrete models of fractional nonlocal continua and fields (for example, see [43–58]). In works [38,39], we propose a lattice fractional calculus, which demonstrates that infiniteness of series of discrete operators is a characteristic property of discretization of derivatives of integer and non-integer orders.

Infinite series of fractional differences and the corresponding derivatives of non-integer orders have been first proposed by Grünwald [27] and by Letnikov [28] in 1897 and 1898 respectively. Now there are other types of fractional differences, which are proposed by Kuttner [29], Cargo and Shisha [30], Diaz and Osler [31], Ortigueira and Coito [32], Ortigueira [33,34], Tarasov [35–37], Ortigueira, Coito and Trujillo [42] and other. The Grünwald–Letnikov differences [1,27,28] and other type of fractional differences, which are proposed in [29–37], cannot be considered an exact discretization of corresponding derivatives, since the difference have unusual characteristic algebraic properties, which do not coincide with the properties of

* Tel.: +7 4959395989.
E-mail address: tarasov@theory.sinp.msu.ru

http://dx.doi.org/10.1016/j.cnsns.2016.01.006
1007-5704/

differential operators. It is easy to see that for integer values of orders, these differences do not have the same algebraic properties as the integer-order derivatives.

It is well-known that the standard finite difference of integer order cannot be considered as an exact discretization of the derivative of this order [86]. The problems of exact discretization of differential equations of integer orders have been formulated in [87–91] (see also [92–94] and [95–99]). Mickens proved that for differential equations there are "locally exact" finite-difference discretization, where the local truncation errors are zero. A main disadvantage of the Mickens discretization of the integer-order derivatives is that these differences strongly depend on the form of the considered differential equation and the parameters of these equations. In addition, the Mickens differences do not have the same algebraic properties as derivative operators of integer orders. A detailed discussion of the Mickens approach is given in Section 2.

Our approach to exact discretization is based on the principle of universality and the algebraic correspondence principle. The differences, which are exact discretization of derivatives, have a property of universality if they do not depend on the form of differential equations and the parameters of these equations. An algebraic correspondence means that an exact discretization of derivatives satisfies the same algebraic relations as the operators of differentiation. In the exact discretization of derivatives, we should have exact discrete analogs of algebraic properties that are performed for derivatives. The exact discrete analog of derivatives should have the same basic characteristic properties as the operators of differentiation. The Leibniz rule is a characteristic property of the derivative operators of integer orders [20,26]. Therefore the exact discretization of these operators must obey this rule. The Leibniz rule should be the main characteristic property of exact discrete analogs of derivative. The second important algebraic property of the exact discretization is the semi-group property. For example, the second-order difference should be equal to the repeated action of the first-order differences. The third important algebraic property of the exact discretization is that the exact difference operators of power-law functions should give the same expression as an action of derivatives.

We propose new approach to exact discretization that is based on new difference operators, which can be considered as an exact discretization of derivatives of integer (and non-integer) orders. These differences do not depend on the form of differential equations and the parameters of these equations. Using these differences, we can get an exact discretization of differential equation of integer and non-integer orders. The suggested approach to exact discretization allows us to obtain difference equations that exactly correspond to the differential equations. We consider not only an exact correspondence between the equations, but also exact correspondence between solutions. We demonstrate that exact discrete analogs of solutions of differential equations are solutions of the corresponding difference equations.

In this paper, a new type of differences of integer and non-integer orders, which are represented by infinite series, are suggested. The suggested differences are derived by discretization of the Riesz differentiation and integration of non-integer and integer orders. The procedure of discretization is proposed by analogy with the procedure of Weyl quantization that preserves the properties of algebraic structures.

In this paper, we propose the differences ${}^{\mathcal{T}}\mathbf{\Delta}^n$, which can be considered as exact discretization of the derivatives d^n/dx^n ($n \in \mathbb{N}$), since these differences preserve the following characteristic property of differential operator.

(1) The Leibniz rule for $\mathcal{T}$-difference

$$ {}^{\mathcal{T}}\mathbf{\Delta}^1 \, (f[n]\,g[n]) = g[n]\, {}^{\mathcal{T}}\mathbf{\Delta}^1 \, f[n] + f[n]\, {}^{\mathcal{T}}\mathbf{\Delta}^1 \, g[n]. \tag{1} $$

(2) The equations of $\mathcal{T}$-difference of power-law functions

$$ {}^{\mathcal{T}}\mathbf{\Delta}^1 \, n^k = k\, n^{k-1}, \quad (k \geq 1). \tag{2} $$

(3) The semigroup property

$$ {}^{\mathcal{T}}\mathbf{\Delta}^1 \, {}^{\mathcal{T}}\mathbf{\Delta}^1 = {}^{\mathcal{T}}\mathbf{\Delta}^2. \tag{3} $$

The suggested differences are represented by infinite series instead of finite series that are the usually used in standard and non-standard (Mickens) differences. The proposed approach to exact discretization allows us to get difference equations that exactly correspond to the differential equations. The suggested discretization of the differential equations is exact for wide class of functions and equations.

2. Problem of exact discretization of equations

There are various approaches to discretization of differential equation with some approximations. In this paper, we do not consider these approached in details. We consider a problem of an exact discretization of the differential equations of integer and non-integer orders. It should be noted that some approaches to exact discretization have been considered in [87–91] (see also [92–94] and [95–97,99]).

The problem of exact discretization has been formulated by Mickens in [89–91]. Mickens proved that for differential equations there are "locally exact" finite-difference discretization, where the local truncation errors are zero.

Let us consider a problem of exact discretization of differential equation and an approach that is suggested by Mickens [89–91]. For simplicity, we consider the first-order ordinary differential equation

$$ \frac{df(x)}{dx} = L_c(f(x), x, \lambda), \quad f(0) = C \quad (x \in \mathbb{R}), \tag{4} $$