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Action-angle variables for the Lie–Poisson Hamiltonian systems associated with Boussinesq equation

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ABSTRACT

Two finite-dimensional Lie–Poisson Hamiltonian systems associated with Boussinesq equation are presented. The action-angle variables in the case of systems with non-hyperelliptic spectral curves are obtained by Sklyanin's method of separation of variables. Moreover, with the help of Hamilton–Jacobi theory for the generating functions of conserved integrals, the Jacobi inversion problems related to the Lie–Poisson Hamiltonian systems and Boussinesq equation are discussed.

1. Introduction

Separation of variables and construction of action-angle variables for finite-dimensional integrable systems generated by 2 × 2 Lax matrices associated with hyperelliptic spectral curves have been studied widely and deeply in the past decades [1–16]. However, the systems generated by 3 × 3 Lax matrices, which are related to non-hyperelliptic spectral curves are much more complicated. Sklyanin in 1992, proposed an efficient method to construct separation variables for the classical integrable $SL(3)$ magnetic chain and $\mathfrak{sl}(3)$ Gaudin model associated with the non-hyperelliptic spectral curves (trigonal curves) [17]. After that, Sklyanin's program is extended to general $SL(N)$ case [18–20]. The method using the algebraic geometric techniques for construction of action-angle variables has appeared in Refs. [21–23], some results for the non-hyperelliptic case were obtained in dealing with the stationary equations of soliton hierarchy [21,22], the $\mathfrak{gl}(N)$ Gaudin model [23] and Jacobi varieties of trigonal curves, from which the solution in terms of multi-variable sigma-functions was obtained [24].

The Boussinesq equation is a typical nonlinear mathematical physics equation associated with a 3 × 3 matrix spectral problem. It was originally introduced in 1871 as a model for one-dimensional weakly nonlinear dispersive water waves propagating in both directions [25]. It has been widely investigated by many authors on various aspects [26–43].

The main purpose of this paper is to construct separation variables and the action-angle variables for the Lie–Poisson Hamiltonian systems associated with Boussinesq equation. We use the method in [17] to construct separation variables and use the Hamilton–Jacobi theory for the generating functions of conserved integrals [15,16] to introduce the action-angle variables.

The outline is as follows. In Section 2, we review the Lie–Poisson structure associated with $\mathfrak{gl}(3,\mathbb{R})$. In Section 3, two Lie–Poisson Hamiltonian systems associated with the Boussinesq equation are presented by using the nonlinearization approach to the adjoint representations of the Boussinesq spectral problem and auxiliary spectral problem. The Lax representation and the

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involution property of conserved integrals are also given by using the generating function method. In Section 4, a reduction of the Lie–Poisson structure on the constraint set of Casimir functions is proposed. It has been proven that the $9N$ dimensional Poisson manifold $(\mathfrak{gl}(3,\mathbb{R})^*)^N$ with Lie–Poisson structure can be reduced to the symplectic manifold $\mathbb{R}^{6N}$ with standard symplectic structure. In Section 5, restricted on the common level sets of Casimir functions, the separable canonical variables $\{\mu_k, \nu_k, k = 1, \dots, 3N\}$ are introduced. In Section 6, based on the Hamilton–Jacobi theory, the generating function for getting the canonical transformation from μ_k, ν_k to action-angle variables I_k, ϕ_k in implicit form is obtained. Further, in terms of the evolution of angle variables, the functional independence of conserved integrals is proven. In addition, the Jacobi inversion problems for the resulting Lie–Poisson Hamiltonian systems and Boussinesq equation are built. A few concluding remarks are given in the last section.

2. Notation and conventions

In this section, we shall introduce some basic notations of Lie–Poisson structure associated with the Lie algebra $\mathfrak{gl}(3,\mathbb{R})$.
Firstly, considering the basis of Lie algebra $\mathfrak{gl}(3,\mathbb{R})$

$$E_{ij} = (\delta_{mi}\delta_{nj}),\ 1 \le i, j \le 3$$

which satisfy the commutative relation

$$[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{li}E_{kj}.$$

Taking the trace form $< A, B >= \mathrm{tr}(AB)$ as a bilinear non-degenerate pairing, we can make an identification $\mathfrak{gl}(3,\mathbb{R}) \cong \mathfrak{gl}(3,\mathbb{R})^*$. In the sense of $< E_{ij}, e_{kl} >= \delta_{ik}\delta_{jl}$, one can find that the dual basis of $\{E_{ij}, 1 \le i, j \le 3\}$ is $\{e_{ij} = E_{ji}, 1 \le i, j \le 3\}$. Hereafter, for the sake of convenience, we choose

$$\mathfrak{gl}(3,\mathbb{R}) = \left\{ \alpha \,\middle|\, \alpha = \sum_{i,j=1}^{3} \alpha_{ij}E_{ji} \right\}, \quad \mathfrak{gl}(3,\mathbb{R})^* = \left\{ y \,\middle|\, y = \sum_{i,j=1}^{3} y^{ij}E_{ij} \right\}.$$

Thus, the corresponding Lie–Poisson bracket for any $F(y), G(y) \in C^\infty(\mathfrak{gl}(3,\mathbb{R})^*)$ is

$$\{F, G\} = \langle y, [\nabla F, \nabla G]\rangle = \mathrm{tr}(y[\nabla F, \nabla G]), \tag{2.1}$$

where the gradient $\nabla F \in \mathfrak{gl}(3,\mathbb{R})$ is defined as

$$\nabla F = \sum_{k,l=1}^{3} \frac{\partial F}{\partial y^{kl}} E_{lk}.$$

Using the cyclicity of the trace, the Hamiltonian vector field associated by (2.1) with a smooth function F is represented by

$$X_F = [\nabla F, y].$$

The structure equations in terms of coordinates $\{y^{ij}, 1 \le i, j \le 3\}$ are

$$\{y^{nm}, y^{lk}\} =< y, [E_{mn}, E_{kl}] >= \delta_{nk}y^{lm} - \delta_{lm}y^{nk},\ 1 \le n, m, l, k \le 3. \tag{2.2}$$

And the three Casimir functions of the Lie–Poisson structure (2.1) are $\mathrm{tr}(y)$, $\mathrm{tr}(y^2)$, $\mathrm{tr}(y^3)$.
In the rest of the paper, we will use the direct product Lie–Poisson bracket on N copies of $\mathfrak{gl}(3,\mathbb{R})^*$:

$$\{F, G\} = \sum_{j=1}^{N} \left\langle y_j, \left[\nabla_j F, \nabla_j G\right]\right\rangle, y_j = \sum_{k,l=1}^{3} y_j^{kl}E_{kl}, \nabla_j F = \sum_{k,l=1}^{3} \frac{\partial F}{\partial y_j^{kl}} E_{lk} \tag{2.3}$$

with the Hamiltonian vector field

$$X_{jF} = \left[\nabla_j F, y_j\right], \quad j = 1, \dots, N$$

and the $3N$ Casimir functions

$$\mathrm{tr}(y_j), \mathrm{tr}\left(y_j^2\right), \mathrm{tr}\left(y_j^3\right), \quad j = 1, \dots, N.$$

3. Lie–Poisson Hamiltonian systems associated with the Boussinesq equation

The Boussinesq equation

$$v_{tt} + \frac{1}{3}v_{xxxx} - \frac{2}{3}(v^2)_{xx} = 0 \tag{3.1}$$

can be obtained by canceling variable u in the following equation