Commun Nonlinear Sci Numer Simulat 30 (2016) 299–306

Contents lists available at ScienceDirect

Commun Nonlinear Sci Numer Simulat

journal homepage: www.elsevier.com/locate/cnsns

Closed-form solutions for the Lucas–Uzawa model of economic growth via the partial Hamiltonian approach

R. Naz [a,*], Azam Chaudhry [b], F.M. Mahomed [c]

[a] Centre for Mathematics and Statistical Sciences, Lahore School of Economics, Lahore, 53200, Pakistan
[b] Department of Economics, Lahore School of Economics, Lahore, 53200, Pakistan
[c] DST-NRF Centre of Excellence in Mathematical and Statistical Sciences, School of Computer Science and Applied Mathematics, University of the Witwatersrand, Johannesburg, Wits 2050, South Africa

ARTICLE INFO

Article history:
Received 30 March 2015
Revised 21 June 2015
Accepted 27 June 2015
Available online 6 July 2015

Keywords:
Economic growth
Partial Hamiltonian approach
Lucas–Uzawa
Current-value Hamiltonian

ABSTRACT

By use of our newly developed methodology (Naz et al., 2014 [1]), for solving the dynamical system of first-order ordinary differential equations (ODEs) arising from first-order conditions of optimal control problems, we derive closed-form solutions for the standard Lucas–Uzawa growth model. We begin by showing how our new methodology yields a series of first integrals for the dynamical system associated with this model and two cases arise. In the first case, two first integrals are obtained and we utilize these to derive closed-form solutions and show that our methodology yields the same results as in the previous literature. In the second case, our methodology yields three first integrals under certain restrictions on the parameters. We use these three integrals to obtain new solutions for all the variables which in turn yield new solutions for the growth rates of these variables. Our results are significant as our approach is applicable to an arbitrary system of ODEs which means that it can also be invoked for more complex models.

1. Introduction

The two-sector endogenous growth model of Lucas and Uzawa is one of the foundations of economic growth theory. Empirical work on economic growth has reinforced the key role of human capital in long run growth (see [2–5] and [6]) and investment in human capital is now a widely accepted component of cross country growth policies.

The basic idea behind the Lucas–Uzawa model ([7,8]) is to find optimal time paths for consumption and the amount of labor devoted to the production of physical capital in an economy which has constrained stocks of physical and human capital. The standard model is based on maximizing a representative agent's utility function subject to constraints on the stock of physical capital and human capital using a current value Hamiltonian system. This model contains two control variables consumption (c) the fraction of labor allocated to the production of physical capital (u) and two state variables, physical capital (k) and human capital (h). One branch of the literature has simplified the model by employing dimension reduction techniques such as the ratio of the original variables, to transform the original system into a lower dimensional more tractable system. Benhabib and Perli [9] and Caballe and Santos [10] analyzed the steady state solutions of by using dimension reduction and then linearization while Mulligan and Sala-i-Martin [11] discussed the transitional dynamics of the reduced model by using a numerical time-elimination method.

* Corresponding author. Tel.: +923315439545; fax: +92426560905.
E-mail address: rehananaz_qau@yahoo.com, drrehana@lahoreschool.edu.pk (R. Naz).

http://dx.doi.org/10.1016/j.cnsns.2015.06.033
1007-5704/

A separate strand has looked at the dynamics of the original variables (see e.g. [12–14]). Xie [12] provided explicit solutions and dynamical properties for the original variables under the parameter restriction that the inverse of the intertemporal elasticity of substitution equals to the elasticity of output with respect to physical capital. At the same time Boucekkine and Ruiz-Tamarit [13] and Hiraguchi [15] gave solutions in terms of hypergeometric functions without relying on parameter restrictions. Later on, Chilarescu [14] proposed a solution procedure using only classical mathematical tools and also compared his results with those derived by Boucekkine and Ruiz-Tamarit [13]. The approach here is distinct.

The objective is to find closed-form solutions for the Lucas–Uzawa model using our newly developed partial Hamiltonian approach (Naz et al [1]) as well as uncovering additional features of the transitional dynamics.

We show how or new methodology yields a series of first integrals and we use these to find closed-form solutions for the Lucas–Uzawa model. We begin by taking two first integrals and show that our approach yields the same results as in the previous literature (see e.g. [13,14,16]). We then show that if we use the three first integrals, we obtain completely new solutions for all the variables in the Lucas–Uzawa model which in turn yield new solutions for the growth rates of these variables. It is worthy to mention here that in this case our partial Hamiltonian methodology yields a solution significantly different from the previous literature. The solutions for the model previously obtained were derived under specific assumptions and were also in terms of integrals which meant that the dynamical system was not completely integrated.

2. Preliminaries

The following definitions are adapted from Naz et al [1] and Dorodnitsyn and Kozlov [17].

Let t be the independent variable and $(q, p) = (q^1, \dots, q^n, p_1, \dots, p_n)$ the phase space coordinates.

Definition 1. The Euler operator $\delta/\delta q^i$ and the variational operator $\delta/\delta p_i$ are defined as

$$\frac{\delta}{\delta q^i} = \frac{\partial}{\partial q^i} - D\frac{\partial}{\partial \dot{q}^i}, \quad i = 1, 2, \dots, n, \tag{1}$$

$$\frac{\delta}{\delta p_i} = \frac{\partial}{\partial p_i} - D\frac{\partial}{\partial \dot{p}_i}, \quad i = 1, 2, \dots, n, \tag{2}$$

where

$$D = \frac{\partial}{\partial t} + \dot{q}_i\frac{\partial}{\partial q_i} + \dot{p}_i\frac{\partial}{\partial p_i} + \cdots \tag{3}$$

is the total derivative operator with respect to t. The summation convention is utilized for repeated indices.

The variables t, q, p are independent and connected only by the differential relations

$$\dot{p}_i = D(p_i), \quad \dot{q}_i = D(q_i), \quad i = 1, 2, \dots, n. \tag{4}$$

Definition 2. The current value Hamiltonian (see, e.g. [1,18]) satisfies

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}^i = -\frac{\partial H}{\partial q^i} + \Gamma_i, \quad i = 1, 2, \dots, n, \tag{5}$$

where Γ_i is a nonzero function of t, p_i, q^i.

Definition 3. The generator

$$X = \xi(t, q, p)\frac{\partial}{\partial t} + \eta^i(t, q, p)\frac{\partial}{\partial q^i} + \zeta_i(t, q, p)\frac{\partial}{\partial p_i}. \tag{6}$$

is a generator of point symmetry of the current value Hamiltonian system (5) if

$$\dot{\eta}^i - \dot{q}^i\dot{\xi} - X\left(\frac{\partial H}{\partial p_i}\right) = 0, \quad \dot{\zeta}_i - \dot{p}_i\dot{\xi} + X\left(\frac{\partial H}{\partial q^i} - \Gamma_i\right) = 0, \quad i = 1, \dots, n \tag{7}$$

are satisfied on the system (5).

Definition 4. An operator X of the form (6) is a partial Hamiltonian operator corresponding to a current value Hamiltonian as in (5), if there exists a function $B(t, q, p)$ such that

$$\zeta_i\frac{\partial H}{\partial p_i} + p_i D(\eta^i) - X(H) - HD(\xi) = D(B) + \left(\eta^i - \xi\frac{\partial H}{\partial p_i}\right)(-\Gamma_i) \tag{8}$$

holds on the solutions of (5).

Note that if H is a present value Hamiltonian, then Eq. (8) becomes the usual determining equation for symmetries of the Hamiltonian action since $\Gamma_i = 0$.

Theorem. *The first integral of system (5) associated with a partial Hamiltonian operator X of the current value Hamiltonian H is determined from*

$$I = p_i\eta^i - \xi H - B, \tag{9}$$

where B(t, p, q) is a gauge function.