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A new solution procedure for the nonlinear telegraph equation

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ABSTRACT

This paper involves a theoretical but fundamental question in the numerical computation of partial differential equations. Is it possible to construct the solution for a *nonlinear* telegraph equation (or a nonlinear damped wave equation) by using a hyperbolic *linear* solution of Klein–Gordon equation? To answer the question, firstly, an analytic solution of the linear Klein–Gordon equation is introduced here. Through the introduction, we show how the original nonlinear telegraph equation can be transformed into an *equivalent* nonlinear system of two integral equations of the second kind. Here, the singularities of the system's kernels are asymptotically shown to be just removable. Then, the above question may be answered by applying Banach fixed point theorem to the two (coupled) integral equations and thus showing how to construct nonlinear iterative solutions of the telegraph equation. This results in a new (functional) iterative procedure for the constructing of the (numerical) solutions of a general nonlinear telegraph equation.

1. Introduction

Our goal here is of a purely theoretical question, however which may be fundamental, especially in the area of computational partial differential equations: Can the hyperbolic *linear* solution-structure of Klein–Gordon equation be directly used to construct the *nonlinear* solution for telegraph equation? For that, we start by reviewing the telegraph equation. It is, also known as a *damped* wave equation, classified as a hyperbolic partial differential equation, which governs physically the voltage and current on an electrical transmission line with distance and time. Much attention has been given in the literature to the development, analysis and implementation for numerical solutions of *special* types of the telegraph equations.

There are challenging issues on *nonlinear* telegraph equations. For example, Fucik and Mawhin [1] studied on generalized periodic solutions of one dimensional nonlinear telegraph equation of the form

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + a\frac{\partial u}{\partial t} + \Phi(u) = f(x,t), \tag{1}$$

where $a > 0$ is a constant, and Φ and f a function of u, and a function of x and t, respectively. Eq. (1) was also investigated on the existence of weak solution of the periodic-Dirichlet problem [2]. Later, Bereanu [3] further examined Eq. (1), using the Leray–Schauder degree theory, existence, nonexistence and multiplicity for the periodic solutions of the nonlinear telegraph equation. El-Azab and El-Gamel [4] presented a new competitive numerical scheme to solve the two dimensional nonlinear telegraph

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equation in the form

$$\frac{\partial^2 u}{\partial t^2} - \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + a\frac{\partial u}{\partial t} + R(x, y, t, u) = 0, \tag{2}$$

in which $a > 0$ denotes a constant, and R an appropriate real-valued function. A full error analysis as well as a numerical experiment was performed, illustrating a good convergence behavior of an approximate solution. The method was based on Rothe's approximation in time discretization and on the Wavelet–Galerkin in the spatial discretization.

Physically, the constant a appearing in Eqs. (1) and (2) implies a damping parameter, involved in a (dissipative) damping effect of wave motion through the term $a \cdot \partial u/\partial t$. As noted above, the damping parameter a in Eqs. (1) and (2) was assumed *constant* [1–4]. However, recently, Van Gorder and Vajravelu [5,6] took into account a *varying* damping in the Nagumo telegraph equation

$$\frac{\partial^2 u}{\partial x^2} - \tau\frac{\partial^2 u}{\partial t^2} - \left[1 + \tau\frac{df}{dt}(u)\right]\frac{\partial u}{\partial t} = f(u), \tag{3}$$

where τ is a constant and f a differentiable function of u. Using Eq. (3), Van Gorder and Vajravelu [5,6] were able to obtain approximate solutions and numerical solutions for some special cases for the boundary value problems via a variational technique. What this paper involves is a (more) general nonlinear hyperbolic telegraph equation, which includes the previous ones as special cases [1–6]

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = F(x,\ t,\ u,\ \partial u/\partial t). \tag{4}$$

Here, the parameter c is a positive constant but F a real valued function of x and t as well as u and $\partial u/\partial t$.

We return to the previous question of how the telegraph equation of Eq. (4) is joined to Klein–Gordon equation. For the question, we first show how the *nonlinear* telegraph equation of Eq. (4) can be transformed into an equivalent nonlinear system of two integral equations of the second kind by introducing a *linear* solution of Klein–Gordon equation in the present study. And then, we apply Banach fixed point theorem to the transformed nonlinear system for developing an iterative procedure to construct the *nonlinear* solutions of the original telegraph equation of Eq. (4). Thus, the question may be answered by proposing a method by the developed iterative procedure for solving the initial value problem of the general nonlinear telegraph Eq. (4), imposed by the initial conditions

$$u(x, 0) = u_0(x),\ \frac{\partial u}{\partial t}(x, 0) = v_0(x). \tag{5}$$

It would be of a new type of (semi-analytic) procedure for constructing the numerical solutions of the hyperbolic nonlinear partial differential equation of Eq. (4). First, we derive the two (coupled) nonlinear integral equations of the second kind, equivalent to the original nonlinear telegraph equation. They may be different from not only the integral equation formulation of BEM (boundary element methods) but the weak formulations of FEM (finite element methods). Second, by applying Banach fixed point theorem to the coupled integral equations, we develop a semi-analytical functional iteration procedure for solving Eq. (4). It is noted here that the process of the proposed procedure depends only on simple usual (numerical) integration. In this respect, there have also been other types of iterative schemes [7–17]. Third, the iteration appears to provide a fast convergence rate for achieving a reasonable solution, as demonstrated in the numerical experiments in this paper. In fact, they show that the method of iteration seems to be quite *simple* and *stable*. This may be because the process of iteration depends only on simple integration and the type of the coupled integral equations, incorporated into the proposed method, is of the *second* kind (second kind integral equations are generally known to have stability properties from the theory of integral equations), as noticed just before. It thus might be one of the useful *nonlinear* methodologies to solve a general nonlinear telegraph equation (for a wide application). Finally, the method may be regarded as a *fully nonlinear* procedure, which does not depend on the assumption of any small parameter as in (asymptotic) perturbation approaches.

2. Method

Klein–Gordon equation, especially well-known in the area of applied mathematics and physical sciences, looks similar to the telegraph equation [17]: e.g., they are classified as the same *hyperbolic* partial differential equations. So, intuitively, they may be closely related to each other. Based on the intuition, in this section, we will show how to construct the *nonlinear* solutions of the telegraph equation by introducing and utilizing Klein–Gordon equation. Prior to discussing a detailed procedure for the construction, we first attempt to transform the telegraph equation into an equivalent system of two nonlinear integral equations. Thereby, an iterative procedure, on the basis of the Banach fixed point theorem [18], is developed for presenting iterative numerical solutions. We begin with Klein–Gordon equation, whose hyperbolic solution structure is embedded into the iterative procedure proposed here, as shall be discussed.

2.1. Introducing Klein–Gordon equation

With a pseudo-parameter, $\alpha > 0$, we add its square times u, i.e., $\alpha^2 \cdot u$, to both sides of Eq. (4),

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} + \alpha^2 u = \tilde{F}, \tag{6}$$