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On the multiscale approximation of solutions to the slowly varying harvested logistic population model

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ABSTRACT

We provide a validation of a formal approximate solution to the problem of the evolution of a slowly varying harvested logistic population. Using a contraction mapping proof, we show that the initial value problem for the population has an exact solution lying in an appropriately small neighbourhood of this approximate solution, under quite general conditions.

1. Introduction

The variation in the population $p(t,\epsilon)$ of a slowly varying harvested logistic model, is described by the solution of the initial value problem [1]:

$$\frac{dp(t,\epsilon)}{dt} = r(\epsilon t)p(t,\epsilon)\left(1-\frac{p(t,\epsilon)}{k(\epsilon t)}\right) - \sigma h(\epsilon t), \qquad p(0,\epsilon) = \mu. \tag{1}$$

Here, $\epsilon > 0, \mu > 0$ and $\sigma \geqslant 0$ are real parameters, with ϵ assumed to be small. Thus, the dependence of r, k and h on ϵt alone indicates that these quantities are slowly varying. It should be noted that the quantities in (1) are dimensionless, so that when the dimensional analogues of r, k and h are constants, we set r, k and h equal to unity, so (1) becomes

$$\frac{dp}{dt} = p(1-p) - \sigma, \quad p(0) = \mu.$$

In the paper referred to above, multiscaling methods (see [2, Ch. 6]) were used to obtain an approximation to the solution of (1) for small positive ϵ, that took the form of a two term expansion

$$p(t,\epsilon) = p_0(t,\epsilon) + \epsilon p_1(t,\epsilon) + O(\epsilon^2), \tag{2}$$

which was shown to agree very well with the results of numerical solution of (1). This construction was purely formal, and was based on the assumption that the initial value problem (1) had a unique solution $p(t,\epsilon)$ that could be represented in this way.

This raises questions about the validity of this formal process and in particular, whether for ϵ sufficiently small and positive, the problem (1) has a unique solution $p(t,\epsilon)$ that satisfies the condition

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$$p(t,\epsilon) - p_0(t,\epsilon) = O(\epsilon), \tag{3}$$

uniformly on an appropriate interval of t values.

We note that (1) is an evolution problem, and so it might be thought logical to seek validation of (3) on $0 \leqslant t < \infty$. However as the asymptotic and numerical solutions have shown [1], the population may in appropriate circumstances, be driven to extinction in finite time; so that in these cases, for physically real representations, we need only consider a finite interval of t values (beyond the first population zero, (1) ceases to be relevant). For situations where the population survives to a nonzero limiting value, $0 \leqslant t < \infty$ will be our choice.

When $\sigma = 0$, (1) reduces to the slowly varying logistic model,

$$\frac{dp}{dt} = r(\epsilon t)p\left(1 - \frac{p}{k(\epsilon t)}\right), \qquad p(0,\epsilon) = \mu, \tag{4}$$

where extinction does not occur, and $0 \leqslant t < \infty$ will be our only possible choice.

Similarly, the density dependent harvesting model

$$\frac{dp(t,\epsilon)}{dt} = r(\epsilon t)p(t,\epsilon)\left(1 - \frac{p(t,\epsilon)}{k(\epsilon t)}\right) - \lambda e(\epsilon t)p(t,\epsilon), \qquad p(0,\epsilon) = \mu, \tag{5}$$

may be rearranged as

$$\frac{dp(t,\epsilon)}{dt} = [r(\epsilon t) - \lambda e(\epsilon t)]p(t,\epsilon)\left(1 - \frac{p(t,\epsilon)}{k(\epsilon t)[1 - \lambda e(\epsilon t)/r(\epsilon t)]}\right), \qquad p(0,\epsilon) = \mu,$$

which takes the form of (4) when $r(\epsilon t) - \lambda e(\epsilon t) > 0$ on $0 \leqslant t < \infty$.

In what follows, we will show, using a contraction mapping argument, that the problem (1) has a unique solution $p(t,\epsilon)$ that satisfies (3) on a finite interval of t values (in cases where extinction occurs) and an infinite interval (in cases where survival to limiting population occurs). The $p_0(t,\epsilon)$ occurring in (3) will be the approximation to the solution of (1) as constructed by multiscaling methods [1]. Our result will be valid under quite general restrictions on the behaviour of the coefficient functions r, k and h, and the significant role the initial value μ plays in (1) will also be made clear. This proof is a refinement of that for the model (1) presented in [3, Ch. 3].

We will then deduce analogous results for the logistic model (4) and harvesting model (5) considered in [4,5] respectively.

We note that Shen et al. [6] adopt a different approach to the slowly varying logistic model (4). By writing $t_1 = \epsilon t$ and with $p(t,\epsilon)$ replaced by $\bar{p}(t_1,\epsilon)$, they convert (4) to

$$\epsilon\frac{d\bar{p}}{dt_1} = r(t_1)\left(1 - \frac{\bar{p}}{k(t_1)}\right), \quad \bar{p}(0,\epsilon) = \mu. \tag{6}$$

They then apply a matched expansions (*fast-slow* time) approach, with a *fast* region adjacent to $t_1 = 0$ and a *slow* region where t_1 is large, to construct an approximation $\bar{p}_0(t_1,\epsilon)$ to the solution of (6). They are then able to prove the existence of an exact solution $\bar{p}(t_1,\epsilon)$ of (6) satisfying the estimate

$$\bar{p}(t_1,\epsilon) - \bar{p}_0(t_1,\epsilon) = O(\epsilon), \tag{7}$$

uniformly on an interval $0 \leqslant t \leqslant T$, for T arbitrarily large and independent of ϵ. They are then able to extend this analysis to obtain analogous results for the harvested model considered here, i.e., (1), on $0 \leqslant t \leqslant T$. In both cases, their proof does not extend to the full interval $0 \leqslant t < \infty$ necessary for such evolution problems.

Subsequently (see [7]), these authors prove an analogous result for the density dependent harvested model (5) (assuming $r(t_1) - \lambda e(t_1) > 0$ on $t_1 \geqslant 0$), with a corresponding estimate (7), uniformly valid on $0 \leqslant t \leqslant T/\epsilon$, again for T arbitrarily large and independent of ϵ. Again, their result does extend to $0 \leqslant t < \infty$.

To our knowledge, no such existence result has been obtained for the harvested model (1).

We begin by making several basic assumptions about the functions r, k and h in (1) in terms of their variation with respect to their argument $t_1 = \epsilon t$ (*slow time*):

A1: The functions $r(t_1), k(t_1)$ and $h(t_1)$ are continuous and continuously differentiable functions of t_1 for all $t_1 \geqslant 0$.

A2: There exist positive constants $\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3, \gamma_1$ and γ_2 which are independent of ϵ such that

$$\alpha_1 \leqslant r(t_1) \leqslant \alpha_2, \qquad \beta_1 \leqslant k(t_1) \leqslant \beta_2, \quad |k'(t_1)| \leqslant \beta_3,$$
$$\gamma_1 \leqslant h(t_1) \leqslant \gamma_2, \quad \text{for all } t_1 \geqslant 0.$$

A3: There exist positive constants ρ_1, ρ_2 independent of ϵ such that the *indicator*

$$\delta(t_1) = 1 - \frac{4\sigma h(t_1)}{r(t_1)k(t_1)},$$

satisfies

$$\delta(t_1) \geqslant \rho_1 \qquad \text{or} \quad \delta(t_1) \leqslant -\rho_2 \quad \text{for all} \quad t_1 \geqslant 0. \tag{8}$$