Commun Nonlinear Sci Numer Simulat 26 (2015) 184–194

Contents lists available at ScienceDirect

Commun Nonlinear Sci Numer Simulat

journal homepage: www.elsevier.com/locate/cnsns

Group classification of nonlinear evolution equations: Semi-simple groups of contact transformations

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ARTICLE INFO

Article history:
Received 25 March 2014
Received in revised form 16 December 2014
Accepted 19 January 2015
Available online 31 January 2015

Keywords:
Group classification
Semi-simple group
Contact transformation

ABSTRACT

We generalize and modify the group classification approach of Zhdanov and Lahno (1999) to make it applicable beyond Lie point symmetries. This approach enables obtaining exhaustive classification of second-order nonlinear evolution equations in one spatial dimension invariant under semi-simple groups of contact transformations. What is more, all inequivalent second-order nonlinear evolution equations which admit semi-simple groups or groups having nontrivial Levi decompositions are constructed in explicit forms.

1. Introduction

In this paper, we study contact symmetries of general second-order nonlinear evolution equations of the form

$$u_t = F(t, x, u, u_1, u_2), \tag{1}$$

where $u = u(t,x), u_t = \partial u/\partial t, u_i = \partial^i u/\partial x^i$ $(i \in \mathbf{N})$, and F is an arbitrary sufficiently smooth real-valued function with $F_{u_2} \neq 0$. In fact, we intend to construct all possible forms of the function F such that the corresponding equation admits nontrivial contact transformation group which contains a semi-simple subgroup. The class (1) includes a number of important and fundamental equations of modern mathematical and theoretical physics, such as the heat, Fisher, Newell–Whitehead and Burgers equations, to mention just a few (see, e.g., [17,22,31]).

Lie group analysis is universally recognized as a versatile and convenient tool for analysis of partial differential equations (PDEs). However, there is a necessary prerequisite for efficient utilization of any group-theoretical method. Namely, the equation under study has to admit a nontrivial Lie group. By this very reason, the problem of group classification of nonlinear PDEs has attracted so much attention and resulted in numerous publications recently.

In the case when a transformation involves dependent and independent variables only, it is called point transformation. For the more general case of transformation including first derivatives of the dependent variables, the term contact transformation has been adopted in the literature. Nowadays, the point group classification of the class (1) has been extensively studied (see [32,3,33] and the references therein). In contrast, much less attention has been devoted to the contact symmetries of the class (1).

The notion of contact (tangential) transformation within the context of differential equations (DEs) was first presented in Sophus Lie's doctoral thesis [15]. He obtained a number of classical results on contact symmetries of ordinary differential

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http://dx.doi.org/10.1016/j.cnsns.2015.01.009
1007-5704/

equations (ODEs). In addition, Lie described linear PDEs in two independent variables admitting contact symmetries [16]. Classification of contact symmetry groups of wave and related equations has been performed in [8]. Contact transformations of nonlinear hyperbolic type equations have been studied in [13,19–21]. The paper [27] tackles the contact symmetry reduction of second-order PDEs. The inter-relationship between contact symmetries and conservation laws has been investigated in [14]. Recently, local conservation laws of second-order evolution equations have been classified up to contact equivalence [25], and the action of contact equivalence transformation on low-order conservation laws of evolution equations of an arbitrary order has been considered in [26]. The concept of discrete transformation group has been modified to include contact transformations in [11].

Sokolov [28,29] dealt with the evolutionary symmetries of the evolution equations

$$u_t = F(x, u, u_1, u_2, \dots, u_n), \quad n \geqslant 2, \tag{2}$$

and Magadeev [18] performed contact group classification of the equations of the form

$$u_t = F(t, x, u, u_1, u_2, \dots, u_n), \quad n \geqslant 2. \tag{3}$$

They obtained a number of nontrivial results on structure and dimension of contact symmetry algebras and described all possible realizations of symmetry algebras of the equations above by Lie vector fields over the field of complex numbers.

What is missing in the research of Sokolov and Magadeev is analysis which of the symmetries listed in [29,18] can be admitted by nonlinear evolution equations in the class (3) of a specific order n. What they have provided is the existence theorem, meaning that for any symmetry presented in [29,18] there exists an order n such that the corresponding PDE (3) can admit the symmetry in question. To utilize the results of [29,18] for complete description of second-order evolution equations admitting contact symmetries, one needs to solve the determining equations (if they are compatible) for each presented Lie algebra realization.

Consequently, description of all possible realizations of admissible symmetry algebras is only part of the group classification of nonlinear evolution equations. Another crucial element of the classification is an actual construction of inequivalent invariant equations of the form (3). Construction of invariant equations has not been considered in the papers [28,29,18] and therefore their solutions of group classification problem of the class (3) is incomplete, to some degree.

One more important point is that the method for computing algebras of contact symmetries developed by Sokolov relies heavily on the fact that an evolution equation by definition involves only the first-order derivative in the temporal variable t. As a result, the transformation law for t involves t only and the variable t enters contact symmetry almost like a parameter. This ensures that contact symmetries of an evolution PDE are closely related to contact symmetries of ordinary differential equations (ODEs) obtained by putting $u_t = 0$. So that efficiency of Sokolov's approach relies on the well-known classification of contact symmetries of ODEs. In fact one can search for finite-dimensional contact symmetries of (3) by considering linear combinations of the generators of contact symmetries of the corresponding ODEs and of the operator ∂_t with coefficients depending on t. In addition, the evolution equations that admit infinite-dimensional contact symmetry groups are known to be linearizable. Evidently, such approach has limited applicability beyond the class of parabolic type PDEs.

In this paper, we develop the alternative approach to classification of contact symmetries of PDEs, which is also a variation of the infinitesimal Lie method. As an application, we perform group classification of second-order evolution equations of the form (1) admitting contact symmetries. We solve completely the problem of constructing all inequivalent realizations of contact transformation groups containing semi-simple sub-groups. Utilizing these results, we construct all Eqs. (1) that are invariant with respect to semi-simple groups or groups containing semi-simple subgroup.

The approach of this paper is the direct generalization of the method used in our papers [3,33] to classify Lie point symmetries of second-order evolution equations. Action of the group of contact transformations imposes stronger equivalence relations than that of point transformations. By this very reason, we get fewer inequivalent realizations than we did for the case of point transformation groups in [3,33]. Any symmetry algebra containing a semi-simple subalgebra obtained in [3,33] is equivalent to one of the realizations derived in this paper with respect to a suitable contact transformation.

If a finite-dimensional algebra L contains the radical N (the largest solvable ideal in L), then due to the Levi–Mal'cev theorem, there exists a semi-simple subalgebra S such that.

$$L = S \Subset N, \tag{4}$$

where S is the Levi factor. The relation (4) is called the Levi decomposition of L. Consequently, any Lie algebra falls into one of the following three categories: (i) semi-simple algebra, (ii) solvable algebra, and, (iii) semi-direct sum of solvable and semi-simple algebra.

Here we restrict our considerations to the cases when symmetry group is either semi-simple or have a nontrivial Levi factor. Note that we have obtained the complete group classification of the PDEs of the class (1) invariant under solvable groups of contact transformations of the dimension $n \leqslant 4$ in [10].

If one adopts the definition of potential symmetry of evolution PDEs suggested by Bluman in [5,6], the following assertion holds. Any potential symmetry of this type admitted by an evolution equation in one spatial dimension can be mapped into a contact symmetry of a related evolution equation of the same order [9,25,26,34]. Consequently, group classification of the class (1) admitting contact symmetries yields, as a by-product, description of PDEs possessing potential symmetries. In other words, it provides nontrivial insight into the largely unexplored world of non-local symmetries of PDEs, since a potential symmetry is a specific case of nonlocal symmetry.