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Solutions for the turbulent classical wake using Lie symmetry methods

A.J. Hutchinson *, D.P. Mason, F.M. Mahomed

School of Computational and Applied Mathematics, Programme for Differential Equations, Continuum Mechanics and Applications, University of the Witwatersrand, Johannesburg, Private Bag 3, Wits 2050, South Africa

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ABSTRACT

We investigate the turbulent planar classical wake and derive the governing equations using the eddy viscosity closure model. The Lie point symmetry associated with the elementary conserved vector is used to generate the invariant solution. We first consider the case where the eddy viscosity depends only on the distance along the wake. We then relax this condition to include the dependence of the eddy viscosity on the perpendicular distance from the axis of the wake. The profiles of the mean velocity show that the role of the eddy viscosity is to increase the effective width of the wake and decrease the magnitude of the maximum mean velocity deficit.

1. Introduction

The wake can be separated into two types: the classical wake and the momentumless wake. The two-dimensional steady flow of the laminar wake of a Newtonian fluid behind a thin symmetric fixed planar body aligned with the mainstream flow, known as a classical wake, was first studied by Goldstein [1]. The two-dimensional classical wake of a shear thinning fluid is investigated in [2]. Similarity reductions are implemented to determine the solutions for the laminar axisymmetric classical wake for power-law fluids in [3]. It was found that the width of the wake is finite for shear thickening flows. A study of the momentumless laminar wake behind a thin symmetric self-propelled body was first investigated by Birkhoff and Zorantello [4]. The two-fluid laminar classical wake was later investigated by Herczynski, Weidman and Burde [5]. The partial differential equation (PDE) for the flow in the wake was derived from the Navier–Stokes equation in the boundary layer approximation. This equation was reduced to an ordinary differential equation (ODE) governing the similarity flow. In this paper we consider the two-dimensional turbulent classical wake.

The turbulent planar wake has been discussed by Tennekes and Lumley [6]. Equations for the mean velocities using the eddy viscosity closure model were formulated. A constant eddy viscosity was chosen. A solution governing the similarity flow was obtained and the results were compared with experimental observations. Existing solutions for the turbulent planar wake are similarity solutions. Similarity solutions can be obtained when the eddy viscosity is a power law of the distance along the axis of the wake and when the kinematic viscosity is neglected. Similarity solutions cannot be obtained for an effective viscosity which is the sum of the kinematic viscosity and the eddy viscosity, and in general when the kinematic viscosity depends on the distance perpendicular to the axis of the wake. In this paper the eddy viscosity closure model is

* Corresponding author.
E-mail addresses: hutchinson.ash@gmail.com (A.J. Hutchinson), David.Mason@wits.ac.za (D.P. Mason), Fazal.Mahomed@wits.ac.za (F.M. Mahomed).

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implemented and the kinematic viscosity is not neglected. The equations for the two-dimensional turbulent classical wake are derived in terms of the stream function. The resulting equations are then solved using Lie point symmetry methods.

Many advances in solving problems in fluid mechanics using symmetry methods have been achieved. A number of the applications of symmetry methods to problems in turbulence can be found in [7–9]. The problem of the wake belongs to the group of problems in fluid mechanics with homogeneous boundary conditions for which a conserved quantity is required for their solution. Another important problem area which requires a conserved quantity to complete the solution is jet flows. Mason [10] for a laminar two-dimensional jet and Ruscic and Mason [11] for a laminar axisymmetric jet applied symmetry methods and derived the group invariant solution using a linear combination of the Lie point symmetries of the equation governing the flow. The conserved quantity and the boundary conditions for the jet were used to solve for the arbitrary constants in the linear combination of Lie point symmetries. The turbulent two-dimensional jet, whose governing equations were formulated using the eddy viscosity closure model, was investigated by Mason and Hill [12]. Again, a linear combination of Lie point symmetries was used to generate the group invariant solution. It was found that the Lie point symmetries only existed provided that the eddy viscosity satisfied a first order linear partial differential equation. Higher order symmetries have been considered in [13,14] and applied to jet flows.

The conserved quantity for the jet plays a central role in the method of solution. The same holds true for the wake. Conserved quantities can be difficult to derive. Much progress has been made recently on deriving conserved quantities using conservation laws for the governing partial differential equations. A systematic approach has been developed to find the conserved quantity for the jet [15]. The conservation laws are first derived and the conserved quantity can then be determined by integrating one of the conservation laws across the jet, chosen to be compatible with the boundary conditions of the problem. Methods to calculate the conservation laws for a partial differential equation can be found in [16]. In this paper we will use the elementary conservation law for the partial differential equation which can be obtained directly from the equation. A modification, due to Kara and Mahomed [17], of the Lie symmetry method was introduced recently for problems with a conserved quantity. They first derived the condition for a Lie point symmetry to be associated with a conserved vector [17,18]. Instead of using a linear combination of all the Lie point symmetries of the partial differential equation to derive an invariant solution, the Lie point symmetry associated with the conserved vector used to derive the conserved quantity was used. This method is more direct. It has been applied to laminar jet flows by Naz and Naeem [19], to a turbulent jet by Mason and Hill [20] and to turbulent flow of a compressible fluid in a tube by Anthonyrajah and Mason [21]. Because a Lie point symmetry associated with a conserved vector of the partial differential equation is used to reduce the partial differential equation to an ordinary differential equation, the ordinary differential equation can be integrated at least once by the double reduction theorem of Sjoberg [22].

In this paper, we consider the governing equation for the two-dimensional turbulent classical wake of a Newtonian fluid using the eddy viscosity closure model. We do not restrict the eddy viscosity to be constant. The Lie point symmetry associated with the elementary conserved vector is used to derive an invariant solution to the problem. The conserved quantity for the classical wake is the drag force [1].

An outline of this paper is as follows. In Section 2, the boundary layer equations for the turbulent wake using the eddy viscosity closure model are derived. In Section 3, the Lie point symmetry associated with the elementary conserved vector is determined. In Section 4, we consider the eddy viscosity to be a function of only the distance along the axis of the wake and solve for the stream function for the turbulent classical wake. Mean velocity profiles are plotted for an eddy viscosity in the form of a power law and the results are compared for a range of power laws and with the laminar classical wake. In Section 5 we consider the eddy viscosity to be a function of the distance along the wake and the perpendicular distance from the axis of the wake. Various forms of the eddy viscosity are investigated and mean velocity profiles are again compared with those obtained for the laminar wake. Finally, conclusions are presented in Section 6.

2. Mathematical model for a two-dimensional turbulent classical wake

Consider a turbulent planar two-dimensional classical wake. An incompressible Newtonian fluid flows past a thin symmetric planar body aligned with the mainstream flow. Cartesian coordinates (x, y) are defined with the trailing edge of the body chosen as the origin as illustrated in Fig. 1.

The flow in the wake downstream of the body is turbulent. It consists of a mean flow and a fluctuation imposed on the mean flow:

$$v_x = \overline{v}_x + v'_x, \quad v_y = \overline{v}_y + v'_y, \quad p = \overline{p} + p', \tag{2.1}$$

where p is the fluid pressure and v_x and v_y are the x- and y-components of the velocity, respectively. The mean flow variables are time averages which are defined as follows [23]:

$$\overline{v}_x(x,y) = \frac{1}{T}\int_t^{t+T} v_x(x,y,t)dt, \quad \overline{v}_y(x,y) = \frac{1}{T}\int_t^{t+T} v_y(x,y,t)dt,$$

$$\bar{p}(x,y) = \frac{1}{T}\int_t^{t+T} p(x,y,t)dt. \tag{2.2}$$