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Noether symmetries and conservation laws of wave equation on static spherically symmetric spacetimes with higher symmetries

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ABSTRACT

In Bokhari et al. (2011), the Noether symmetries of the wave equation on a general spherically symmetric spacetime metric were provided in terms of five functions of the variables t and r subject to certain differential equations and constraints. In this work, these differential equations and constraints are solved for all the static spherically symmetric spacetimes admitting G_{10} or G_7 or G_6 as maximal isometry group. As a result, the Noether symmetries and the corresponding conservation laws for all static spherically symmetric spacetimes admitting higher symmetries are obtained explicitly.

1. Introduction

A complete classification of static spherically symmetric Lorentzian manifolds according to their isometries and metrics was obtained in [2,3] where it was shown that these admit G_{10} or G_7 or G_6 or G_4 as maximal isometry group. In a recent work [4], a study of the Lie symmetries of the wave equation on static spherically symmetric spacetimes, admitting G_{10} or G_7 or G_6, is carried out. The aim of this paper is to investigate the Noether symmetries and conservation laws of the wave equation on these spaces; hence completing the symmetry analysis of the wave equation on static spherically symmetric spacetimes admitting higher symmetries. The standard procedure for determining the conservation laws for the variational systems is given by the well known Noether theorem [5]. This theorem requires a Lagrangian. There are approaches that do not require a Lagrangian or even assume the existence of a Lagrangian for differential equations. These approaches include direct construction methods for multipliers [6], partial Lagrangian [7] and the new conservation theorem [8].

For the case of wave equation on spherically symmetric spacetimes, Bokhari et al. [1] provided the Noether symmetries in terms of five functions of the variables t and r subject to certain differential equations and constraints. Here we consider the wave equation on static spherically symmetric spacetimes admitting G_{10} or G_7 or G_6 and completely solve the differential equations and constraints of [1] to obtain the corresponding Noether symmetries and conservation laws. For references related to the applications and various approaches of obtaining conservation laws, the reader is referred to [9–13].

Using the d'Alembert's operator $\Box_g u = \frac{\partial}{\partial x_i}\left(\sqrt{|g|}g^{ik}\frac{\partial u}{\partial x_k}\right)$, the wave equation $\Box_g u = 0$ on spherically symmetric spacetime with the metric

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$$ds^2 = e^{\nu(r,t)}dt^2 - e^{\lambda(r,t)}dr^2 - e^{\mu(r,t)}d\theta^2 - e^{\mu(r,t)}\sin^2\theta d\varphi^2 \tag{1.1}$$

can be written as

$$\frac{\partial}{\partial t}\left(m_1\frac{\partial u}{\partial t}\right) + \frac{\partial}{\partial r}\left(m_2\frac{\partial u}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(m_3\frac{\partial u}{\partial \theta}\right) + \frac{\partial}{\partial \phi}\left(m_4\frac{\partial u}{\partial \phi}\right) = 0 \tag{1.2}$$

where $m_1 = e^{\left(\mu-\frac{\nu}{2}+\frac{\lambda}{2}\right)}\sin\theta, m_2 = -e^{\left(\mu+\frac{\nu}{2}-\frac{\lambda}{2}\right)}\sin\theta, m_3 = -e^{\left(\frac{\nu}{2}+\frac{\lambda}{2}\right)}\sin\theta$ and $m_4 = -e^{\left(\frac{\nu}{2}+\frac{\lambda}{2}\right)}\csc\theta$.

Throughout the paper we use a convention in which derivatives of u with respect to t, r, θ and ϕ are respectively represented by $u_1,\ u_2,\ u_3$ and u_4. In addition, we will use the notation

$$X = [\xi_1, \xi_2, \xi_3, \xi_4, \eta]$$

to represent the Noether symmetries

$$X = \xi_1\frac{\partial}{\partial t} + \xi_2\frac{\partial}{\partial r} + \xi_3\frac{\partial}{\partial \theta} + \xi_4\frac{\partial}{\partial \phi} + \eta\frac{\partial}{\partial u}$$

of the wave Eq. (1.2).

The Lagrangian of the wave Eq. (1.2) is given by the expression

$$2L = m_1u_1^2 + m_2u_2^2 + m_3u_3^2 + m_4u_4^2. \tag{1.3}$$

Section 2 provides the simplified system that can be used to determine the Noether symmetries of the wave Eq. (1.2) on static spherically symmetric spacetimes in an efficient manner. The conservation laws of angular momentum and energy for the general case are also given here. The system of Section 2 is solved in Sections 3–5 to respectively obtain the Noether symmetries of the wave Eq. (1.2) on all static spherically symmetric spacetimes admitting $G_{10},\ G_7,\ G_6$ as maximal isometry group. Employing the Noether theorem, the corresponding conservation laws are also obtained in Sections 3–5.

2. Noether symmetries of wave Eq. (1.2) on static spherically symmetric spacetimes

As mentioned in Section 1, the system and the constraints that can be used to determine the Noether symmetries of the wave equation on general spherically symmetric spacetimes were provided in [1]. Introducing the auxiliary function

$$Z = (k_2\sin\phi - k_1\cos\phi)\sin\theta - k_7\cos\theta \tag{2.1}$$

in the results of [1], the Noether symmetries of the wave equation on static spherically symmetric spacetimes in terms of the functions $k_1(t,r),\ k_2(t,r),\ k_7(t,r),\ k_8(t,r),\ k_9(t,r)$ and the three constants $d_1,\ d_2,\ d_3$ can be given as

$$\begin{aligned}
\xi_1 &= e^{\mu-\nu}Z_t + k_8\\
\xi_2 &= -e^{\mu-\lambda}Z_r + k_9\\
\xi_3 &= Z_\theta + d_2\sin\phi - d_1\cos\phi\\
\xi_4 &= Z_\phi\csc^2\theta + (d_1\sin\phi + d_2\cos\phi)\cot\theta + d_3\\
\eta &= \alpha(t,r,\theta,\phi)u + \beta(t,r,\theta,\phi)
\end{aligned} \tag{2.2}$$

where

$$\alpha = -e^{\mu-\nu}Z_{tt} - \frac{1}{2}e^{\mu-\nu}(2\mu_t - \nu_t)Z_t + \frac{1}{2}e^{\mu-\lambda}\nu_rZ_r - \frac{1}{2}(\nu_tk_8 + \nu_rk_9 + 2k_{8,t}) \tag{2.3}$$

and β is an arbitrary solution for the wave equation. The corresponding gauge terms are given as [1]

$$B_1 = \frac{1}{2}\sin\theta\, e^{\mu-\frac{\nu}{2}+\frac{\lambda}{2}}(\alpha_tu^2 + 2\beta_tu) \tag{2.4}$$

$$B_2 = -\frac{1}{2}\sin\theta\, e^{\mu+\frac{\nu}{2}-\frac{\lambda}{2}}(\alpha_ru^2 + 2\beta_ru) \tag{2.5}$$

$$B_3 = -\frac{1}{2}\sin\theta\, e^{\frac{\nu}{2}+\frac{\lambda}{2}}(\alpha_\theta u^2 + 2\beta_\theta u) \tag{2.6}$$

$$B_4 = -\frac{1}{2\sin\theta}e^{\frac{\nu}{2}+\frac{\lambda}{2}}(\alpha_\phi u^2 + 2\beta_\phi u) \tag{2.7}$$

So the problem of determining the Noether symmetries is reduced to finding five functions of two variables (t and r), namely, $k_1,\ k_2,\ k_7,\ k_8,\ k_9$ by first solving the following system [1]