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Self-adjointness and conservation laws of difference equations

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ABSTRACT

A general theorem on conservation laws for arbitrary difference equations is proved. The theorem is based on an introduction of an adjoint system related with a given difference system, and it does not require the existence of a difference Lagrangian. It is proved that the system, combined by the original system and its adjoint system, is governed by a variational principle, which inherits all symmetries of the original system. Noether's theorem can then be applied. With some special techniques, e.g. self-adjointness properties, this allows us to obtain conservation laws for difference equations, which are not necessary governed by Lagrangian formalisms.

1. Introduction

Symmetries of variational principles are naturally symmetries of the associated Euler–Lagrange equations. A connection between such type of symmetries and conservation laws for differential equations is established via Noether's theorem [3,9,12,16,22,23]. However, Noether's theorem has difficulty in applications to arbitrary differential equations. In [2], this was overcome by linearizing a given differential system and then constructing conservation laws via the provided formula, while the adjoint invariance condition should be satisfied. In [13], the author defined an adjoint system for arbitrary differential equations, and hence a Lagrangian for a given differential system together with its adjoint system. Symmetries of a given differential system can then be extended to variational symmetries for the Lagrangian. At this stage, Noether's theorem is applicable.

Geometric methods, especially symmetry analysis, as are used for investigating differential equations, have been applied to difference equations over the last decades, see [11,17,30]. A discrete version of Noether's theorem exists, see for example [5,8,19,25]. In the present paper, we attempt to generalize the direct method in [13] for constructing conservation laws from differential systems to difference systems. For a system of difference equations, its transformation groups of symmetries can be extended to groups of variational symmetries for a Lagrangian governing the original system itself together with its adjoint system. Thus, Noether's theorem can be used to construct conservation laws for the combined system. For (strict, quasi, weak) self-adjoint systems, it is possible to transfer such conservation laws to conservation laws of the original system. This procedure can also be realized once special solutions of the adjoint system are known.

2. Direct construction of conservation laws for differential equations

Let $x = (x^1, x^2, \dots, x^p)$ and $u = (u^1, u^2, \dots, u^q)$ be p independent variables and q dependent variables, respectively. Let $J = (j_1, j_2, \dots, j_p)$, and let u^α_J denote $|J|$th-order partial derivatives of u. Here $|J| = j_1 + j_2 \cdots + j_p$ and

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$$u_J^\alpha := \frac{\partial^{|J|} u^\alpha}{(\partial x^1)^{j_1} (\partial x^2)^{j_2} \cdots (\partial x^p)^{j_p}}. \tag{1}$$

Consider a linear differential operator (the Lie–Bäcklund operator),

$$\begin{aligned} X &= \xi^i D_i + \left(\eta^\alpha - \xi^i u_i^\alpha\right) \frac{\partial}{\partial u^\alpha} + \cdots + \sum_{\alpha,J} D_J \left(\eta^\alpha - \xi^i u_i^\alpha\right) \frac{\partial}{\partial u_J^\alpha} + \cdots \\ &= \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \cdots . \end{aligned} \tag{2}$$

Here $\xi^i = \xi^i(x, [u])$ and $\eta^\alpha = \eta^\alpha(x, [u])$ are smooth functions, where $[u]$ denotes u and its derivatives. The operator D_i is the total derivative with respect to x^i and D_J is a composite of total derivatives. The set of all such differential operators is a Lie algebra equipped with the usual Lie bracket between two vector fields.

For a system of partial differential equations

$$F_\alpha(x, [u]) = 0, \quad \alpha = 1, 2, \ldots, q, \tag{3}$$

the adjoint system is given by

$$0 = F_\alpha^*(x, [u], [v]) := \mathbf{E}_{u^\alpha}\left(v^\beta F_\beta\right). \tag{4}$$

The Euler operator $\mathbf{E}_{u^\alpha}$ is defined as

$$\mathbf{E}_{u^\alpha} := \frac{\partial}{\partial u^\alpha} - D_i \frac{\partial}{\partial u_i^\alpha} \cdots + (-D)_J \frac{\partial}{\partial u_J^\alpha} + \cdots, \tag{5}$$

where $(-D)_J = (-1)^{|J|} D_J$. The system of differential equations (3) and (4) corresponds to a Lagrangian $L(x, [u], [v]) = v^\alpha F_\alpha(x, [u])$, and we have

$$\begin{aligned} &\mathbf{E}_{u^\alpha}(L) = F_\alpha^*(x, [u], [v]), \\ &\mathbf{E}_{v^\alpha}(L) = F_\alpha(x, [u]). \end{aligned} \tag{6}$$

A differential system is said to be self-adjoint if the system $F_\alpha^*(x, [u], [u]) = 0$ is identical to the original one.

If a differential operator $X = \xi \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha}$ is a symmetry generator for Eqs. (3), there always exists a variational symmetry generator for the Lagrangian L, namely $Y = \xi \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \eta_*^\alpha \frac{\partial}{\partial v^\alpha}$. The coefficient η_*^α is determined by the generator X. Hence, by using Noether's theorem and the generator Y, we can construct conservation laws for the combination of (3) and (4). With knowledge of particular solutions of v or self-adjointness of the original system, we can get conservation laws of the original Eqs. (3). For more details, consult [13] (see also [2]).

Example 2.1 [13]. Consider the Korteweg–de Vries equation

$$u_t = u_{xxx} + u u_x \tag{7}$$

and its adjoint equation

$$v_t = v_{xxx} + u v_x. \tag{8}$$

It is easy to see that the KdV equation is self-adjoint. These two equations are governed by the following Lagrangian

$$L = v(u_t - u u_x - u_{xxx}). \tag{9}$$

Symmetries of a scaling transformation with generator

$$X = -3t \frac{\partial}{\partial t} - x \frac{\partial}{\partial x} + 2u \frac{\partial}{\partial u} \tag{10}$$

will be extended to variational symmetries of the Lagrangian L, that is,

$$Y = -3t \frac{\partial}{\partial t} - x \frac{\partial}{\partial x} + 2u \frac{\partial}{\partial u} - v \frac{\partial}{\partial v}. \tag{11}$$

Conservation law $D_t P^1 + D_x P^2$ obtained through Noether's theorem is hence given by

$$P^1 = (3t u_{xxx} + 3t u u_x + x u_x + 2u) v \tag{12}$$

and

$$P^2 = -(2u^2 + x u_t + 3t u u_t + 4u_{xx} + 3t u_{txx}) v + (3u_x + 3t u_{tx} + x u_{xx}) v_x - (2u + 3t u_t + x u_x) v_{xx}. \tag{13}$$

Setting $v = u$ and transferring the terms of the form $D_x(\ldots)$ from P^1 to P^2, we get

$$P^1 = u^2, P^2 = u_x^2 - 2u u_{xx} - \frac{2}{3} u^3. \tag{14}$$