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Early-warning signs for pattern-formation in stochastic partial differential equations

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ABSTRACT

There have been significant recent advances in our understanding of the potential use and limitations of early-warning signs for predicting drastic changes, so called critical transitions or tipping points, in dynamical systems. A focus of mathematical modeling and analysis has been on stochastic ordinary differential equations, where generic statistical early-warning signs can be identified near bifurcation-induced tipping points. In this paper, we outline some basic steps to extend this theory to stochastic partial differential equations with a focus on analytically characterizing basic scaling laws for linear SPDEs and comparing the results to numerical simulations of fully nonlinear problems. In particular, we study stochastic versions of the Swift–Hohenberg and Ginzburg–Landau equations. We derive a scaling law of the covariance operator in a regime where linearization is expected to be a good approximation for the local fluctuations around deterministic steady states. We compare these results to direct numerical simulation, and study the influence of noise level, noise color, distance to bifurcation and domain size on early-warning signs.

1. Introduction

Drastic sudden changes in dynamical systems, so-called critical transitions or tipping points, occur in a wide variety of applications. It is often desirable to find early-warning signs to anticipate transitions in order to avoid or mitigate their effects [75]. There has been tremendous recent progress in determining potential warning signs in various sciences such as ecology [77,82], climate science [58,59], engineering [20,61], epidemiology [55,68], biomedical applications [65,83] and social networks [54]; see also [74,76] for concise overviews. For a large class of critical transitions, the underlying dynamical mechanism involves a slow drift of a system parameter towards a local bifurcation point, where a fast transition occurs [49]. This class has been referred to as "B-tipping" in [2]. A detailed mathematical analysis of the underlying stochastic fast-slow systems, including their generic scaling laws, can be found in [51]; see also [5] for further mathematical background.

An example of a warning sign occurring in many stochastic systems is an increase in variance as a bifurcation point is approached [15]. This effect is intrinsically generated by critical slowing down (or "intermittency" [36,78]), i.e. the underlying deterministic dynamics becoming less stable near the bifurcation point. Hence, (additive) stochastic fluctuations become dominant approaching a B-tipping point.

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A substantial effort has been made to extract early-warning signs, such as slowing down and variance increase, from univariate time series e.g. using various time series analysis methods [39,60,62], normal forms [80], topological methods [7] and generalized models [57]. Although theoretical tests and models with sufficiently large data sets tend to work very well [23,51], there are clear limits to predictability [12], particularly when relatively sparse data sets are considered [17,25,27,56].

For systems with spatio-temporal dynamics (and associated spatio-temporal data), the additional data in the spatial direction may be used to improve existing early-warning signs and to discover new ones. If a system is initialized in a spatially patterned state instead of a homogeneous one, then measures of the pattern could be considered as potential candidates to provide warning signs. For example, in [46] the patchiness of states in a vegetation model is used. However, for a uniform homogeneous steady state that undergoes a bifurcation, such warning signs are not expected to be available.

Many early-warning signs computed for univariate time series have multivariate time series analogs, such as spatial variance and skewness [28,37] as well as slowing down and spatial correlation [24]. An "averaging" over the spatial direction, e.g. in the sense of the Moran coefficient [26], can be helpful to facilitate direct comparisons with univariate indicators. Also, a natural alternative to avoid the full complexity of spatio-temporal pattern formation is to focus on early-warning signs for traveling waves [52]. Despite these exploratory works, it is quite clear at this point that the full mathematical analysis of early-warning signs for stochastic spatio-temporal systems is largely uncharted territory. Furthermore, a better theoretical understanding of spatio-temporal warning signs will significantly improve practical multivariate time series analysis, which is one of the main motivations for this study.

For finite-dimensional B-tipping, a quite robust classification scheme [2,49] has been formulated and associated warning signs have been investigated (up to generic codimension-two bifurcations) based upon normal forms, fast-slow systems and stochastic analysis [51]. Such a detailed scheme is much more difficult to develop for spatio-temporal systems since there is no complete generic bifurcation theory for all spatio-temporal systems available. However, it is expected that certain subclasses of stochastic partial differential equations (SPDEs) have warning-signs near pattern-forming bifurcations that can be studied in detail. For deterministic partial differential equations (PDEs), quite a number of bifurcations leading to pattern-formation are well studied; for example, see [21,22,43] and references therein.

Translating and extending the *qualitative* pattern-forming results from PDEs to SPDEs is an extremely active area of current research. We refer to [9,32] for additional background and references. However, when searching for early-warning signs, it is important to augment the qualitative results with *quantitative* scaling laws.

There has been a lot of interest recently in early-warning signs for spatio-temporal systems.[2] Furthermore, early-warning signs for particular models have been studied in the context of modeling case studies. For example, measures of one-point temporal variance and correlation for spatio-temporal processes, which track temporal statistics at one spatial point, are natural extensions of the generic early-warning signs developed for univariate time series [74]. Previous studies have computed these measures in moving windows for various two-dimensional (2D) processes generated by simulations in which a parameter drifts slowly in time, finding that signals of impending transitions are sometimes obscured by past measurements in the moving window [28,26]. For a process generated by a pattern-forming vegetation model, however, autocorrelation at lag 1 is found to increase monotonically approaching a Turing bifurcation [24].

More robust signals of critical transitions in spatio-temporal processes are expected to lie in explicitly spatial measures. Such measures computed at points in time, in contrast with temporal measures computed in a moving window, reflect the instantaneous (rather than residual) state of a system [37]. Spatial variance, skewness, correlation length, and patchiness have previously been shown to increase before a sudden transition for various 2D processes [37,28,26,24].

Although it may seem intuitively clear that the classical warning signs from SODEs should also be found in SPDEs on bounded domains, there is no complete mathematical theory available to address how classical early-warning signs can be generalized from stochastic ordinary differential equations (SODEs) to SPDEs. In this paper, we limit ourselves to several elementary steps working toward this generalization:

- (R1) We review the available literature from various fields. In particular, there are closely related contributions from statistical physics, dynamical systems, stochastic analysis, theoretical ecology and numerical analysis.
- (R2) We outline the basic steps to generalize classical SODE warning signs, such as autocorrelation and variance increase, to the spatio-temporal setting motivated by two standard models for pattern formation, the Swift–Hohenberg (SH) equation and the Ginzburg–Landau (GL) equation. In particular, we focus on a regime before the bifurcation, where the linearization around the homogeneous state is expected to provide a very good approximation to local stochastic fluctuations. The main result is a scaling law of the covariance operator before bifurcation from a homogeneous branch.
- (R3) We numerically investigate the SH and GL equation to connect back to several spatio-temporal warning signs proposed in applications, particularly in the context of ecological models. We compare the numerical results for the nonlinear systems to the analytical results obtained from linear approximation in (R2). The numerical results reveal two distinct scaling regimes. Furthermore, we obtain several additional numerical results about the influence of several natural parameters (domain size, distance to bifurcation, noise level and noise correlation length) on early-warning signs.

[2] For example, at the two recent workshops: (I) "Critical Transitions in Complex Systems" at Imperial College London, 19–23 March, 2012; (II) "Tipping points: fundamentals and applications" at ICMS Edinburgh, 9–13 September, 2013.