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Nonlinear oscillator with power-form elastic-term: Fourier series expansion of the exact solution

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ABSTRACT

A family of conservative, truly nonlinear, oscillators with integer or non-integer order nonlinearity is considered. These oscillators have only one odd power-form elastic-term and exact expressions for their period and solution were found in terms of Gamma functions and a cosine-Ateb function, respectively. Only for a few values of the order of nonlinearity, is it possible to obtain the periodic solution in terms of more common functions. However, for this family of conservative truly nonlinear oscillators we show in this paper that it is possible to obtain the Fourier series expansion of the exact solution, even though this exact solution is unknown. The coefficients of the Fourier series expansion of the exact solution are obtained as an integral expression in which a regularized incomplete Beta function appears. These coefficients are a function of the order of nonlinearity only and are computed numerically. One application of this technique is to compare the amplitudes for the different harmonics of the solution obtained using approximate methods with the exact ones computed numerically as shown in this paper. As an example, the approximate amplitudes obtained via a modified Ritz method are compared with the exact ones computed numerically.

1. Introduction

Nonlinear oscillations in physics, engineering, mathematics and related fields have been the subject of intensive research for many years and several methods have been used to find approximate solutions to these dynamical systems [1,2]. In conservative nonlinear oscillators the restoring force is not dependent on time, the total energy is constant [2,3] and any oscillation is stationary. An important feature of the solutions for conservative oscillators is that they are periodic and range over a continuous interval of initial values [4]. Conservative truly nonlinear oscillatory systems are modeled by differential equations for which the restoring force has no linear approximation at $x = 0$ [4].

In this paper we consider a class of conservative truly nonlinear oscillators with an odd power-form elastic-term, $f(x) = \operatorname{sgn}(x)|x|^{\alpha}$, where $\alpha \geqslant 0$. In recent years some examples of this class of truly nonlinear oscillators have been analyzed. Mickens [5] studied the oscillations in a $x^{4/3}$ potential and Hu and Xiong [6] extended this study to a more general

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$x^{(2m+1)/(2n+1)}$ potential, where m and n are arbitrary non-negative integers. Several techniques have been used to obtain analytical approximate solutions, such as harmonic balance, parameter expansion, iteration or averaging methods [4–15]. Gottlieb [8] analyzed the frequencies of oscillators with fractional-power nonlinearities introducing the sign function which enables the sign of the restoring force to be changed. He also obtained an expression for the exact period of this type of oscillators. Pilipchuk [16] studied a more general class of these oscillators in which the exponent α of x continuously takes any non-negative real value, such as odd, even, rational or irrational. Pilipchuk obtained an approximate solution as a series expansion and found that this solution is more accurate when the exponent α increases. A new analytical method for solving the differential equations, which describe the motion of the oscillator with fraction order elastic force, was introduced by Cveticanin [14]. In this technique the Krylov–Bogolubov method was extended. By considering a variable frequency, Kovacic and Rakaric [17] developed a procedure to obtain higher-order approximations for oscillators with a fractional-order restoring force. They adjusted the Ritz method by introducing an approximate Lagrangian and using the exact value of the frequency of oscillations they obtained explicit expressions for the amplitudes of the second- and third-order approximations. Cveticanin [18] analyzed the vibrations of oscillators with non-integer order nonlinearity and time variable parameters. Cveticanin and Pogány [19] studied free and self-excited vibrations of oscillators with polynomial nonlinearity. Recently, Elías-Zúñiga and Martínez-Romero [20] obtained accurate solutions for a generalized power-form elastic term oscillator using the enhanced cubication method developed by these authors. In an interesting paper, Muñoz and Fernández-Anaya [21] showed that this family of conservative truly nonlinear oscillators with an odd power-form elastic-term arose from particular solutions of Abel's mechanical problem [22].

For oscillators in which only one odd power-form elastic-term exists, the exact analytical period can be obtained in terms of Gamma functions using the energy integral [14] and the explicit solution can be expressed as a cosine-Ateb function [19,23], which represents the inversion of the incomplete Beta functions. However, only a few examples of this type of oscillators – such as linear, anti-symmetric constant force, pure-quadratic and pure-cubic oscillators – have closed-form solutions [4]. However, it will be shown in this paper that it is possible to obtain the Fourier series expansion of the exact solution for all of these conservative truly nonlinear oscillators, even though their exact solutions are unknown. The coefficients of the Fourier series expansion of the exact solution are obtained as a function of the order of nonlinearity (the exponent α) and are computed numerically. Since the nonlinear restoring force is an odd function of the displacement x, only the odd harmonics appear in the Fourier series expansion of the periodic solution. We compare the coefficients computed using this method with those obtained for the Fourier series expansion for the exact solution both for the anti-symmetric constant force oscillator and the pure-cubic oscillator. We present some examples to illustrate the usefulness of the technique for obtaining approximate solutions with a finite number of harmonics by truncating the Fourier series expansion. One of the possible applications of this method is to compare the amplitudes for the different harmonics of the solution obtained using approximate methods with the exact ones computed numerically as shown in this paper. As an example, we compare the approximate amplitudes obtained via a modified Ritz method [17] with the exact ones computed numerically. As may be seen, this allows us to determine the range of values for the order of nonlinearity for which the approximate Lagrangian used gives accurate results.

2. Formulation and solution procedure

We consider a class of conservative single-degree-of-freedom nonlinear oscillators modeled by the second-order autonomous differential equation

$$\frac{d^2x}{dt^2} + \mathrm{sgn}(x)|x|^{\alpha} = 0 \tag{1}$$

with initial conditions

$$x(0) = A, \quad \frac{dx}{dt}(0) = 0 \tag{2}$$

where x and t are the non-dimensional displacement and time, respectively, and the order α of nonlinearity is any positive real number ($\alpha \geqslant 0$). These oscillators are truly nonlinear oscillators [4] in which the nonlinear function $f(x) = \mathrm{sgn}(x)|x|^{\alpha}$ is odd, i.e. $f(-x) = -f(x)$ and satisfies $xf(x) > 0$ for $x \in [-A, A]$, $x \neq 0$, where A is the oscillation amplitude. The system oscillates around the equilibrium position $x = 0$ and the period, T, and periodic solution, x, are dependent on α and A.

Introducing a new reduced variable, $u = x/A$, we can rewrite Eqs. (1) and (2) as

$$\frac{d^2u}{dt^2} + A^{\alpha-1}\mathrm{sgn}(u)|u|^{\alpha} = 0 \tag{3}$$

$$u(0) = 1, \quad \frac{du}{dt}(0) = 0 \tag{4}$$

We consider $0 \leqslant t \leqslant T/4$, where T is the period of the oscillations. For these values of t we have $0 \leqslant u \leqslant 1$. Integration of Eq. (3) gives the first integral