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A novel variational formulation for thermoelastic problems

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ABSTRACT

A novel variational formulation for thermoelasticity is proposed in this paper. The formulation is based on the Hamilton–Pontryagin principle and the concept of temperature displacement. Although there are many other papers that have a similar goal, most of the proposed approaches are quite complicated, and contain assumptions that curtail their applicability. The proposed variational principle in this paper is straightforward with no extra assumptions and it is in conformity with the Clausius–Duhem inequality as a statement of the second law of thermodynamics. Conservation laws for linear momentum and energy, and the constitutive equation for thermoelasticity are consequences of this variational formulation.

1. Introduction

A variational description of a physical system consists of a statement that the variation of a specified functional is equal to some fixed value, which can be customarily chosen to be zero. Attempts to state variational principles for natural laws date back in history and the development of such variational principles have received attention due to their elegance and the advantages they exhibit when solving practical problems. Various variational formulations for thermomechanical problems have been suggested both for discrete and continuous systems in the past decades and authors have usually introduced new variables and quantities in stating the principle from their respective viewpoints.

Many of the pioneering works in this direction have been due to M.A. Biot, who has published many significant papers in this area [1–7]. In his variational principle, Biot introduced a quantity called the *entropy displacement vector* $\bar{S}$, in addition to the common displacement vector $\bar{u}$, to describe the thermal part of his formulation. This quantity is defined by the following equation,

$$\frac{\partial \bar{S}}{\partial t} = \frac{1}{\theta_r}\frac{\partial \bar{H}}{\partial t}, \tag{1}$$

where $\frac{\partial \bar{H}}{\partial t}$ is the rate of the heat flow $\bar{H}$, and θ_r is the temperature of the environment, that is assumed to be constant. He also introduced a non-negative quadratic dissipation function in terms of generalized velocities that is proportional to the entropy production. From these, he obtained a variational formulation, which yielded Euler–Lagrange equations that

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Nomenclature

$i, j, k, \ldots$	indices in continuum mechanics
$\mathbf{b}$	Body force per unit volume
$\mathbf{e}$	the strain tensor
$\mathbf{p}$	the momentum vector
$\mathbf{p}_u$	mechanical momentum
$\mathbf{p}_\tau$	thermal momentum
$\mathbf{q}$	thermomechanical generalized coordinate vector
$\mathbf{t}$	the stress tensor
$\mathbf{u}$	mechanical displacement vector
$\mathbf{v}$	the velocity vector
$\mathbf{w}$	mechanical velocity vector
r	rate of heat supply
t	time
$\mathbf{F}$	deformation gradient tensor
$\bar{S}$	entropy displacement vector
$\bar{H}$	heat flow
$\hat{\mathbf{K}}$	heat conductivity tensor
$\mathbf{Q}$	heat flux vector
$Q_{i,i}$	divergence of heat flux vector
$\hat{\mathbf{P}}$	first Piloa stress tensor
θ	instantaneous temperature
θ_0	initial temperature
θ_r	temperature of environment
$\acute{\theta}$	dimensionless temperature
τ	temperature displacement
ρ_0	the density in reference configuration
$\hat{\eta}$	entropy function
$\hat{\psi}$	Helmholtz free energy

describe thermomechanical systems [6]. Biot's formulation was applied to heat conduction [8], and some nonlinear problems in heat transfer [9].

Kermidas and Ting [10] used temperature-based variables instead of entropy displacement. They assumed a linear relation between entropy and a dimensionless temperature that is defined to be

$$\acute{\theta} = \frac{\theta - \theta_0}{\theta_0}, \tag{2}$$

where θ and θ_0 are instantaneous and initial absolute temperatures, respectively. These linear constitutive relations limit the applicability of the approach to a local region of validity.

Some authors have suggested the futility of developing variational formulations for thermomechanical problems and have proposed the use of other methods, like Galerkin projection, for approximating thermomechanical systems [11]. He et al. [12] obtained a variational principle for coupled thermoelasticity with finite displacement using the semi-inverse method for the field equations directly. To do this, they replaced the time derivative terms in the coupled heat conduction equations with finite-difference approximations. Sawada [13] derived a variational principle for nonlinear and non-steady (non-equilibrium) thermodynamic systems using the principle of maximum entropy production. He also applied this approach to simulate a chemical structure with a growing random pattern [14]. Maugin and Kalpakides [15] formulated a variational principle based on the inverse motion mapping and then explored the corresponding Euler–Lagrange equations. Subsequently, they also derived a Hamiltonian formulation from the Lagrangian formulation. Yang et al. [16] developed a variational formulation for general dissipative solids, where they made a distinction between the external temperature and the equilibrium temperature. Apostolakis and Dargush [17] used the mixed variational principle for thermoelastic materials. Their resulting relations are only valid for problems in the linear regime because they assumed that the temperature was constant in the energy equation.

In the present work, a new variational formulation for thermoelastic problems is proposed. This formulation contains no *a priori* assumptions which limit its validity except for the most commonly accepted assumptions in thermoelasticity. In the next section, we will review the necessary constraints on the thermomechanical responses to ensure that the second law of thermodynamics is satisfied. In addition, the energy equation for thermoelasticity is also stated. In the third part, the Lagrangian and Hamiltonian for a thermomechanical problem are derived. In the fourth section, the Hamilton–Pontryagin principle is presented and finally, the variational formulation is proposed in the last section using this principle.