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Enthalpy balance methods *versus* temperature models in ice sheets ☆

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ABSTRACT

In this paper we propose and numerically solve an original enthalpy formulation for the problem governing the thermal behaviour of polythermal ice sheets. Although the modelling follows some ideas introduced in Aschwanden and Blatter (2009), nonlinear basal boundary conditions in both cold and temperate regions are also considered, thus including the sliding effects in the frame of a fully coupled shallow ice approximation (SIA) model. One of the main novelties of this work comes from the introduction of the Heaviside multivalued operator to take into account the discontinuity of the thermal diffusion function at the cold-temperate transition surface (CTS) free boundary. Moreover, we propose a duality method for maximal monotone operators to solve simultaneously the nonlinear diffusive term and the free boundary. Some numerical simulation examples with real data from Antarctica are presented and illustrate the small differences between the computed results from the enthalpy formulation here proposed and the alternative formulation in terms of the temperature (Calvo et al., 2001).

1. Introduction

Greenland and Antarctic ice sheets evolutions have been recently pointed out as tipping elements in the global climate change due to their complex interactions with the rock-beds, the atmosphere, the oceans and the sea-level rise, among others (see Lenton et al. [25] and references therein). For instance, an important goal in contemporary Earth science is to relate mass and energy-balance states of the large ice sheets to observed sea-level rise. A measure of these states could provide an accurate quantification of the ice-sheet system response to climate change. So, it is of fundamental importance to focus on improving ice-sheet models, including representation of key processes and nonlinear transitions. Nowadays, ice-sheet models are no longer constrained to use overly simplified physics, allowing them to simulate accurately thermal, mechanical and dynamical behaviors to make more realistic projections for the next few centuries. In the context of energy balance models for ice sheets the thermodynamic behavior of ice flow can be treated by using cold-ice models or polythermal ones.

In glaciology, the polythermal ice masses are defined as those ones containing both cold and temperate ice separated by a cold-temperate transition surface (CTS). Cold ice is below the pressure melting point temperature while temperate ice is at pressure melting point temperature. Basal layers of temperate ice appear in glaciers located in the Canadian Arctic and also

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in Greenland and Antarctica. Inside the temperate region liquid water can be generated by dissipation heating. Therefore, the so called *cold-ice models* based on a differential equation for the temperature (see [27], for example) cannot take into account the full energy components in the temperate region of polythermal glaciers and fail to be energy preserving. More precisely, changes in the latent heat content in temperate ice are not taken into account in the temperature state variable considered in cold ice models (see [2], for detailed explanations). *Polythermal models* overcome this drawback by decomposing the ice mass into two disjoint cold and temperate regions, the CTS being a free boundary separating both domains. In this setting, one additional advantage of the enthalpy method is that the CTS can be obtained as a level set of the enthalpy variable, thus allowing the use of fixed domain formulations and avoiding the numerical disadvantages related to alternative front tracking techniques, for example.

From the modelling point of view, it is important to notice that the temperature is the relevant thermodynamical magnitude in the cold ice region while in the temperate ice region the water fraction is the relevant magnitude (as the temperature is constant and equal to melting one). Mixture theory has been used in the literature to explain polythermal ice masses (see Fowler and Larson [17], Hutter [21], Greve [18,19] and Fowler [16], among others). In the recent work by Aschwanden and Blatter [1] an enthalpy formulation (referred as *enthalpy gradient method*) to model both the cold and temperate regions is proposed. For this purpose, the heat flux is expressed in terms of the enthalpy gradient and the enthalpy is related to the temperature in the cold region and to the water content in the temperate region. More precisely, in [1] a discontinuous diffusive coefficient at the unknown CTS appears in the energy equation for the enthalpy. Also a regularization procedure taken from sea ice modelling literature (brine pocket parametrization scheme) to approximate this discontinuous coefficient is proposed. However, in [1] the flow is not thermomechanically coupled and a prescribed ice velocity field is considered. More recently, in [2] the mixture theory is used to obtain a relationship of the enthalpy with the water content and temperature without this regularization. Additionally, a fully coupled model is considered.

Enthalpy methods are frequently used in computational fluid dynamics, specially to model Stefan problems describing phase change moving boundaries, such as in this study involving melting processes. It can be remembered that Joseph Stefan compared his calculations for the melting of the polar ice cap with the existing observational data around 1890. From there, Stefan problems were soon found to be important in many other areas of the natural sciences. For instance, Stefan problems occur in geophysical problems related to magma dynamics, border movement in sedimentary basins, permafrost, sea ice, among others. In industrial processes, the Stefan problem arises in metal solidification, food freezing, ice production and many others. And elsewhere, as cryosurgery problems in medical science, where cancer cells may be destroyed under extremely cold temperatures. But, in ice sheet modelling enthalpy formulations are novel a few years ago and could be very useful to glaciologists because they require only a single energy balance equation for the entire domain consisting of co-existing solid, liquid and mushy states (cold ice, water, temperate ice). Thus, in the enthalpy reformulation the governing equation, incorporating the latent heat, stays the same for any phase and the energy balance conservative property of the system is directly preserved. Intermediate states and the phase change region are determined from the equilibrium liquid volume fraction *versus* temperature relationship, thereby avoiding the necessity for a front tracking numerical method to solve the thermal problem. Hence, the formulation is ideal for studying phase change of the evolving ice sheet.

An additional advantage of enthalpy models, already pointed out in [1], consists of the CTS is a function of the enthalpy variable. No explicit surface-representation scheme is required and no *a priori* restrictions apply to CTS shape. In contrast, enthalpy is a nontrivial function of temperature, water content, and pressure.

One of the innovative aspects of the present work comes from the enthalpy formulation of the energy equation in terms of a multivalued Heaviside operator for the diffusive coefficient which avoids the regularization approximation proposed in [1]. In many topics related to thermal problems with phase change it is well known that a suitable physical formulation involves the use of a multivalued enthalpy. Actually, the variation of the enthalpy at the phase change temperature is not instantaneous because the latent heat is not consumed or released instantaneously. So, the enthalpy jump at the melting point temperature is produced by the product of the latent heat and density. As the enthalpy is actually unknown at melting point temperature, then we formulate such a jump by means of a multivalued Heaviside operator instead of a switch function. In this setting, the added mathematical difficulty is solved by means of some knowledge about subdifferential calculus and maximal monotone operators [4]. In fact, the strong nonlinearity associated to the Heaviside operator is solved by means of a duality type method, also used to solve the moving boundary aspect involved in the cold-temperate transition surface. However, the numerical algorithm becomes simple because it uses classical Yosida approximations of the operator (see [3]) that have been already efficiently implemented and validated by the authors (and many others) in previous works.

Although the formulation based on the Heaviside operator is here introduced in the shallow ice setting, this idea can be applied to other mathematical models related to climate. For example, this is the case of some models appearing in [24], where a nonlinear switch parametrization accounts for the representation of the free boundary between dry and moist regions in the frame of a general circulation model.

Once the previous enthalpy formulation has been introduced, in order to simplify the conservation and constitutive equations governing ice sheet thermodynamics, the shallow ice approximation (SIA) can be considered. For this purpose, SIA has been frequently employed in the literature to simulate ice sheets dynamics over realistic domains (Greenland, Antarctica) and over entire glacial cycles (see [20,14,15,6], among others). Thermocoupled shallow ice models have been around since the work [22] and different attempts to solve them appear in the literature (see [23], for example). After the previous works [8–11], recently in [12] the authors propose and solve numerically a fully thermocoupled SIA model. More precisely, after a suitable initialization of magnitudes, at each time step of the global algorithm the ice profile is updated by solving an