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Stability of the synchronized network of Hindmarsh–Rose neuronal models with nearest and global couplings

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ABSTRACT

We analytically and numerically investigate a set of N identical and non-identical Hindmarsh–Rose neuronal models with nearest-neighbor and global couplings. The stability boundary of the synchronized states is analyzed using the Master Stability Function approach for the case of identical oscillators (complete synchronization) and the Kuramoto order parameter for the disordered case (phase synchronization). We find that, through a linear coupling modeling electrical synapses, complete synchronization occurs in a system of many nearest-neighbor or globally coupled identical oscillators, and in the case of non-identical neurons it is stable even in the presence of a spread of the parameters. We find that the Hindmarsh–Rose neuronal models can synchronize when coupled through the action of potential variable or through the interaction by rapid flows of ions through the membrane. The degree of connectivity of the network favors synchronization: in the global coupling case, the threshold for the in-phase state stabilizes when the number of dynamical units increases. The transition from disordered to the ordered state is a second order dynamical phase transition, although very sharp.

1. Introduction

Complex networks based on the interplay between complexity in the overall topology and local dynamical properties of the coupled units, have proved a challenging task for the study of synchronization of dynamical units. Nevertheless, synchronization of coupled oscillators can be nowadays tackled in a unified framework for different phenomena [1,2]. Neurological systems are involved in several kinds of synchronization such as complete synchronization [3–6], lag synchronization [7,8], phase synchronization [9,10] and quasi-synchronization [11,12]. In the study of synchronization, it is important to determine the stability of the synchronous state of a generic network topology, and hence a generic coupling configuration. Consequently, powerful mathematical methods are being developed to seek the potential conditions for realization of the most interesting form of dynamical behavior, such as synchronization, that can arise in these networks. These methods reverse the question by studying when a synchronous state is stable, in terms of coupling schemes and of coupling strengths. The Master Stability Function (MSF) approach has been introduced to address this question for arrays of coupled oscillators [13,14]. The method has been extended to the case of complex networks of dynamical systems coupled with arbitrary topologies [15–22]. Quite naturally, the neuronal activity has emerged as a prominent example of network dynamics. Neurologically, the

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synchronous state is usually interpreted as intensity keeping of the neuronal response during the propagation over the network. The study of perfect synchronization among identical models is just preliminary to the study of realistic disordered models where perfect synchronization is impossible. However, the study of spiking-bursting synchronization from biophysical models may be helpful to understand further the information processing in the brain [6].

The Hindmarsh–Rose (HR) complex dynamical system is a well-known model of a single neuron. In this model, the action potentials firing, due to a sufficient stimulus, induces several behavior modes, and the system exhibits a deterministic chaos for an appropriate choice of some neuronal parameters [23,26–30]. Moreover, it is known that two HR neurons can synchronize when coupled by the action potential variables. Recently, Daliborg [31] reported on exponential fast synchronization of two HR neurons with novel unidirectional coupling. It has been shown, through the Lyapunov function, that a single parameter bound is a sufficient criterion for the global stability of synchronous state. Multi-neuron cases have also been considered, when disorder afflicts a single parameter [32,33].

The chaotic activity suggests for the HR neuronal model is of a particular interest for the understanding of collective dynamical behaviors. In the ideal case, a network of neurons consists of identical components. However, the neurons do not have the same physical or chemical characteristics: many of the neuronal parameters are weakly modified by small perturbations [28]. Thus, if the network of identical HR neuronal models could be considered in a first approximation, a refined model requires to consider non-identical neurons. Generally (and qualitatively) one considers two kinds of synapses: electrical and chemical. Corson et al. [28,35] have recently studied the synchronization of HR neuronal model by an average term proportional to the synaptic influence.

In this paper, we consider a network of HR neuronal models with nearest and global couplings. The coupling between neurons is produced by the action potential variables [31,28,35], that model the interactions due to the rapid flow of ions through the membrane. The coupling between neurons through the slow flows variables is not consider here, inasmuch as we want to investigate the neurons during the firing activity.

In the case of a network of identical coupled neurons, the stability of synchronization is studied by the MSF [13], which displays the emergence of some instabilities in the collective behavior, like de-synchronization phenomena [15] or spatial temporal chaos. For the case of non-identical coupled neurons, the synchronization process is investigated using the Kuramoto order parameter [36–38], which assimilates a neuron in the network to an oscillator having its own phase and amplitude.

The paper is organized as follows: After the description of the HR model in Section 1, in Section 2 we describe the network and the MSF approach, that allows to analyze the stability of the synchronization process in the network. In the same Section, we check the results obtained from the MSF approach with direct numerical simulations. We present in Section 4 the network of HR neuronal models in the presence of a spread of parameters. Thereafter, we study the condition of phase synchronization using the Kuramoto order parameter. Section 5 deals with conclusions.

2. The Hindmarsh–Rose neuronal model

The HR model [23] of neuronal activity is aimed to study the spiking-bursting behavior of the potential membrane experimentally observed in a single neuron. In fact the original Hodgkin–Huxley equations cannot describe bursting and are only able to reproduce spiking. In fact, the model was improved in 1984 [23] taking in account slow currents to describe bursting. Other systems can also describe the neuronal bursting activity as the Rulkov map [24] and the Huber–Braun model [25]. Concerning HR model, the relevant variable is the membrane potential, $x(t)$ (in dimensionless units). There are two more variables, $y(t)$ and $z(t)$, which take into account the transport of ions across the membrane through the ion channels. The transport of sodium and potassium ions occurs through fast ion channels, and its rate is described by $y(t)$, the spiking variable. The transport of other ions occurs through slow channels, and is taken into account through $z(t)$, the bursting variable. The corresponding mathematical model is a system of three first order nonlinear ordinary differential equations for the dimensionless dynamical variables $x(t), y(t)$, and $z(t)$. They read:

$$\begin{cases} \dot{x} = y - x^3 + ax^2 - z + I, \\ \dot{y} = 1 - dx^2 - y, \\ \dot{z} = r[s(x - x_1) - z]. \end{cases} \tag{1}$$

Here, the parameter r governs the different time scale between the fast and the slow dynamics. The applied current I, which discriminates normal physiological neuronal activity, models the fluxes of ionic charges across the cytoplasm occurring when a sufficient stimulus (that is in fact an electrical potential) is applied to the neuron. The fluxes can be controlled during the experiments to leap out their influence on the action potential on the neuronal activity. The $z(t)$ flux is a slow current whose rate is of the order of the small parameter r $(0 < r < 1)$ that governs the bursting and adaptation behavior of the model, while s only governs adaptation. Through this study, we set the parameter of the 3D HR neuronal model as follows [28,39]: $a = 3$, $d = 5$. The equilibrium of the two-dimensional HR system, $x_1 = -\frac{1}{2}\left(1 + \sqrt{5}\right)$, corresponds to the threshold potential to trigger bursts. The 3D HR neuronal model is known to exhibit many types of robust activities that are generic for most models derived from the original Hodgkin–Huxley equations. It also allows for some regulation of the bursting activity that is referred as the square-wave bursting [27,40,41] in neuronal models [42–49]. Neuronal activity is a mixture of two phases: (1) The rest phase, when the neuron does not emit any action potential (also known as the slow neuronal