ARTICLE IN PRESS

Commun Nonlinear Sci Numer Simulat xxx (2014) xxx–xxx

Contents lists available at ScienceDirect

Commun Nonlinear Sci Numer Simulat

journal homepage: www.elsevier.com/locate/cnsns

Using general quadratic Lyapunov functions to prove Lyapunov uniform stability for fractional order systems

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ARTICLE INFO

Article history:
Received 18 August 2014
Accepted 13 October 2014
Available online xxxx

Keywords:
Fractional calculus
Uniform stability of fractional order systems
Fractional extension of Lyapunov direct method
General quadratic Lyapunov functions
Fractional adaptive systems

ABSTRACT

This paper presents two new lemmas related to the Caputo fractional derivatives, when $\alpha \in (0, 1]$, for the case of general quadratic forms and for the case where the trace of the product of a rectangular matrix and its transpose appear. Those two lemmas allow using general quadratic Lyapunov functions and the trace of a matrix inside a Lyapunov function respectively, in order to apply the fractional-order extension of Lyapunov direct method, to analyze the stability of fractional order systems (FOS). Besides, the paper presents a theorem for proving uniform stability in the sense of Lyapunov for fractional order systems. The theorem can be seen as a complement of other methods already available in the literature. The two lemmas and the theorem are applied to the stability analysis of two Fractional Order Model Reference Adaptive Control (FOMRAC) schemes, in order to prove the usefulness of the results.

1. Introduction

Fractional calculus relates with the calculus using integrals and derivatives of orders that may be real or complex. It has become very popular in recent years due to its demonstrated applications in many fields of science and engineering [1].

Fractional operators can be found in the identification field, when modeling systems using fractional differential equations, such as the diffusion process found in batteries [2], some heat transfer process [3], the effect of the frequency in induction machines [4], amongst others.

Fractional operators can be found in the control field as well; e.g. Fractional Order PID controllers [5]; fractional high gain output feedback controllers [6,7] and Fractional Order Model Reference Adaptive Controllers (FOMRAC) [8–14], amongst others.

The stability of these systems have to be proved using techniques specially developed for fractional order systems (FOS), either for the stability of the system itself, or for the stability of the controlled system.

One of the available techniques to prove the stability of FOS is the fractional-order extension of Lyapunov direct method, proposed by Li et al. [15]. This method allows concluding asymptotic stability and Mittag–Leffler stability for FOS. However,

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http://dx.doi.org/10.1016/j.cnsns.2014.10.008
1007-5704/

Please cite this article in press as: Duarte-Mermoud MA et al. Using general quadratic Lyapunov functions to prove Lyapunov uniform stability for fractional order systems. Commun Nonlinear Sci Numer Simulat (2014), http://dx.doi.org/10.1016/j.cnsns.2014.10.008

it does not address the frequent case when the fractional derivative of the Lyapunov function is only negative semidefinite, and conclusions about stability or uniform stability can be drawn. The work reported in [16] can be seen as a complement of the method proposed in [15], since it allows proving stability in the Lyapunov sense when the fractional derivative of the Lyapunov function is only negative semidefinite, although the case of uniform Lyapunov stability is not addressed in [16]. Nevertheless, the use of those methods is often a hard task, since finding Lyapunov functions is more complex in the fractional order case than in the integer order case.

Recently, a new lemma for the Caputo fractional derivative of a quadratic function has been presented in [17]. This result allows the use of classic quadratic Lyapunov functions in many stability analysis of FOS. However, in some cases, those simple quadratic functions are not useful, and more general quadratic Lyapunov functions must be used instead. Furthermore, in many cases the use of the trace of matrices inside Lyapunov functions can be useful in proving the stability of FOS, but there is no well established results in these cases. This paper presents a new lemma related to the Caputo fractional derivatives, when $\alpha \in (0, 1]$, for general quadratic functions, which allows using general quadratic Lyapunov functions in the stability analysis of many FOS. Besides, another lemma related to the Caputo fractional derivative of the trace of a product of matrices is presented, which allows using the trace inside Lyapunov functions, for stability analysis of FOS.

The paper is organized as follows; Section 2 presents some basic concepts about fractional calculus and stability of FOS, as well as some well known results that are used in this paper. Section 3 introduces the new lemmas for Caputo fractional derivatives of special functions, and also a theorem related to the fractional extension of Lyapunov direct method, for proving uniform stability in the Lyapunov sense. Section 4 presents the usefulness of the proposed lemmas and the theorem for the stability analysis of two FOMRAC schemes. Finally, Section 5 presents the conclusions of the work.

2. Preliminary concepts

In this section, some basic definitions related to fractional calculus and some concepts and techniques related to the stability analysis of FOS are presented.

2.1. Fractional calculus

In fractional calculus, the traditional definitions of the integral and derivative of a function are generalized from integer orders to real or complex orders. In the time domain, the fractional order derivative and fractional order integral operators are defined by a convolution operation.

Several definitions exist regarding the fractional derivative of order $\alpha > 0$, but the Caputo definition in (1) is used in most of the engineering applications, since this definition incorporates initial conditions for $x(\cdot)$ and its integer order derivatives, i.e., initial conditions that are physically appealing in the traditional way.

Definition 1 (*Caputo Fractional Derivative [1]*). The Caputo fractional derivative of order $\alpha \in \mathbb{R}^+$ on the half axis $\mathbb{R}^+$ is defined as follows

$$ {}_{t_0}^{C}D_t^{\alpha}x(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} \frac{x^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \quad t > t_0 \tag{1} $$

with $n = \min\{k \in \mathbb{N}/k > \alpha\},\ \alpha > 0$.

A new property for Caputo fractional derivatives has been presented in [17], and is introduced here for completeness in Lemma 1.

Lemma 1 ([17]). *Let $x(t) \in \mathbb{R}^n$ be a differentiable vector. Then, for any time instant $t \geqslant t_0$*

$$ \frac{1}{2}\,{}_{t_0}^{C}D_t^{\alpha}\left[x^T(t)x(t)\right] \leqslant x^T(t)\,{}_{t_0}^{C}D_t^{\alpha}x(t), \quad \forall \alpha \in (0,1) \tag{2} $$

2.2. Stability of fractional order systems

Regarding the stability analysis of FOS, the fractional-order extension of the Lyapunov direct method [15] is one of the available methodologies in the literature, and is stated in Theorem 1.

Using the Caputo derivative, a FOS can be defined in a general form as

$$ {}_{t_0}^{C}D_t^{\alpha}x(t) = f(x,t) \tag{3} $$

In this study we will consider $\alpha \in (0, 1)$.

Definition 2. A continuous function $\gamma : [0, t) \to [0, \infty)$ is said to belong to class-K if it is strictly increasing and $\gamma(0) = 0$ [15].

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