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Circular, elliptic and oval billiards in a gravitational field

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ABSTRACT

We consider classical dynamical properties of a particle in a constant gravitational force and making specular reflections with circular, elliptic or oval boundaries. The model and collision map are described and a detailed study of the energy regimes is made. The linear stability of fixed points is studied, yielding exact analytical expressions for parameter values at which a period-doubling bifurcation occurs. The dynamics is apparently ergodic at certain energies in all three models, in contrast to the regularity of the circular and elliptic billiard dynamics in the field-free case. This finding is confirmed using a sensitive test involving Lyapunov weighted dynamics. In the last part of the paper a time dependence is introduced in the billiard boundary, where it is shown that for the circular billiard the average velocity saturates for zero gravitational force but in the presence of gravitational it increases with a very slow growth rate, which may be explained using Arnold diffusion. For the oval billiard, where chaos is present in the static case, the particle has an unlimited velocity growth with an exponent of approximately 1/6.

1. Introduction

In the 1920s billiards were introduced by Birkhoff [1] into the theory of dynamical systems. They consist of a point particle moving freely in a region except for specular collisions with the boundary. Birkhoff's idea was to have a simple class of models which shows the complicated behavior of non-integrable smooth Hamiltonian systems without the need to integrate a differential equation [1,2]. Depending on the shape of the boundary, billiard dynamics may be (i) regular, with only periodic or quasi-periodic orbits present; (ii) mixed, in which chaos, KAM islands (also called periodic islands) and invariant spanning curves that limit the chaos in the systems are present; (iii) completely ergodic, presenting only chaos in the phase space.

In recent years billiards continue to provide useful models for Hamiltonian dynamics, as well as problems involving free motion in cavities and in extended structures. The latter includes the study of anomalous diffusion in geometries with infinite horizon [3–5]. Mixed phase space models include mushroom billiards [6,7]. Time irreversible billiards have also been considered [8] as well as connections with many wave and quantum mechanical problems [9–12]. Further examples are reviewed in [13].

It can be useful to open the billiard geometry by including a hole and investigating escape time and related distributions. Open billiards provide a useful starting point for an understanding of more general classes of open dynamical systems [13]

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and include the study of open circular billiards and the Riemann hypothesis [14]. Sometimes it is useful to investigate the survival probability or the histogram of particles that reached certain height in the phase space [15].

An important subject in this area is the study of Fermi acceleration (FA) [16]. The phenomenon is characterized by an unlimited energy growth of a bouncing particle undergoing collisions with a periodically moving and heavy wall. According to the Loskutov–Ryabov–Akinshin (LRA) conjecture [17], the introduction of a time dependence to the boundary of a billiard is a sufficient condition to observe FA when the corresponding static billiard has chaotic components. The elliptical case however must be treated separately and the LRA conjecture does not apply for it. For the static boundary it is integrable and hence has a phase space showing only regular structures. There are two quantities which are preserved in the elliptical billiard with static boundary: (i) energy (E) and; (ii) product of the angular momenta about the two foci (F) as discussed in [18]. However, it was shown recently [19] that the introduction of a periodically time perturbation to the boundary does lead to FA. The explanation for observing diffusion in velocity is mainly related to the existence of a separatrix in the phase space. According to [19], after introducing a time perturbation to the boundary, the separatrix turns into a stochastic layer yielding a type of turbulent behavior in F therefore leading to diffusion in velocity, hence producing the FA. More simulations were done in the model [20] which confirmed the FA. Due to this observation in the elliptical billiard and considering the LRA conjecture, a recent work [21] argues that the existence of heteroclinic fixed point in the phase space may extend the conjecture in the absence of chaos in the phase space. The circular billiard does not have such a fixed point and remains regular even with vibrating boundaries. Even in [21], the authors claim that FA seems not to be a robust phenomena. The reflection law may be modified so that the particle experiences a slightly inelastic collision therefore having a fractional loss of energy upon collision. Even in such a small limit of dissipation, the unlimited energy growth is suppressed.

It is also interesting to study billiards in the presence of a constant gravitational field. The Galton board (1873) is a mechanical device that exhibits stochastic behavior. It consists of a vertical (or inclined) board with interleaved rows of pegs, where a ball moving into the Galton board moves under gravitation and bounces off the pegs on its way down [22]. There have been many recent investigations in gravitational billiards including a physical experiment observing stable islands in chaotic atom-optics billiards [23], characterization of the dynamics of a dissipative, inelastic gravitational billiard [24], the study of linear stability in billiards with potential [2] and many others [25–28]. It also includes the wedge billiard [24,29–31] which has singular regions in the phase space that cannot be described by the KAM theorem [26]. In Ref. [2] it is discussed the way to obtain the linear stability of billiards with potentials for which the free motion is integrable. Examples include the linear gravitational potential, the constant magnetic field, the harmonic potential, and a billiard in a rotating frame of reference, imitating the restricted three-body problem [2].

The purpose of the present paper is to investigate some gravitational convex billiards, with or without vibrating boundaries. As noted above, the static and vibrating circular billiards and static elliptical billiards have regular dynamics in the absence of a gravitational field, while oval and vibrating elliptical billiards usually have mixed phase space. We apply the approach in Ref. [2] to characterize linear stability of fixed points in the presence of the external field exactly in terms of the curvature of the boundary and normal component of the velocity at the point(s) of collision. A numerical search identifies apparently ergodic energy values, even for the circular billiard; we then apply a sensitive Lyapunov weighted dynamics test to further confirm this [32]. As mentioned in the work of Tailleur and Kurchan [32], this method was proposed and adapted from the context of chemical reactions [33–35]. They have demonstrated that their method is capable of finding even very small stability regions in systems of many degrees of freedom [36]. Ergodicity is interesting as it may suggest the existence of a new class of ergodic billiards, extending known results for dispersing and defocusing non-gravitational billiards on one hand and for the gravitational but piecewise linear wedge billiard on the other. In non-gravitational billiards it is known that no smooth convex billiard can be ergodic [37,38].

Finally, we introduce a time-dependence in the billiard, and show for the circle in the absence of gravitational field and as expected, the average velocity of the system approaches a regime of saturation. In the presence of gravitational field, the velocity keeps growing for long number of collisions but with a small slope of growth, an interesting effect since in the high velocity limit the gravitational field has less and less effect, so the dynamics approaches the non-gravitational regular behavior. We can explain the continued but slow acceleration in terms of Arnold diffusion. For the breathing oval billiard, our numerical result for the slope of growth is slightly larger than the one obtained in the [39] and theoretically foreseen in [40] but still of the same order of magnitude.

This paper is organized as follows: in the Section 2 we describe the model and obtain the mapping that describes the dynamical of a particle. In Section 3 we study the energy regimes and explore the phase space for different values of the control parameters. Section 4 is devoted to study some periodic orbits, for the oval, ellipse and consider the low energy regime. Apparent ergodicity is studied in the Section 5. In Section 6 we take into account the time-dependent billiards. Our conclusions and final remarks are presented in 7.

2. The model and the map

The models we are considering consist of a classical particle (or an ensemble of non-interacting particles) confined inside and experiencing collisions with a closed boundary of circular, elliptic and oval shapes under the presence of a gravitational force. To describe the dynamics, we follow the same general procedure as made in [39]. Then the dynamics of each particle is described in terms of a four-dimensional nonlinear mapping $T(\theta_n, \alpha_n, |\vec{V}_n|, t_n) = (\theta_{n+1}, \alpha_{n+1}, |\vec{V}_{n+1}|, t_{n+1})$ where the variables