Commun Nonlinear Sci Numer Simulat 22 (2015) 1036–1049

Contents lists available at ScienceDirect

Commun Nonlinear Sci Numer Simulat

journal homepage: www.elsevier.com/locate/cnsns

From a generalised Helmholtz decomposition theorem to fractional Maxwell equations

Manuel D. Ortigueira [a,*], Margarita Rivero [b], Juan J. Trujillo [c]

[a] *UNINOVA and DEE of Faculdade de Ciências e Tecnologia da UNL[1], Campus da FCT da UNL, Quinta da Torre, 2829-516 Caparica, Portugal*
[b] *Departamento de Matemática Fundamental, Universidad de La Laguna, 38271 La Laguna, Tenerife, Spain*
[c] *Departamento de Análisis Matemático, Universidad de La Laguna, 38271 La Laguna, Tenerife, Spain*

A R T I C L E I N F O

Article history:
Received 30 July 2014
Received in revised form 4 September 2014
Accepted 7 September 2014
Available online 16 September 2014

Keywords:
Grünwald–Letnikov
Fractional gradient
Fractional divergence
Fractional curl
Fractional laplacian

A B S T R A C T

The main objective of this paper is to propose a new generalisation of the Helmholtz decomposition theorem for both fractional time and space, which leads to four equations generalising the Maxwell equations that emerge as particular case. To get these results the well-known classical vectorial operators, gradient, divergence, curl, and laplacian are generalised to fractional orders using Grünwald–Letnikov approach.

© 2014 Elsevier B.V. All rights reserved.

1. Introduction

1.1. On the fractional world

Fractional Calculus was born at the same time of the integer order Calculus [26,34,47,65]. However only in the last thirty years left the Mathematical ghetto and started invading the application world, namely Physics and Engineering. This delay may be due to the lack of simple geometrical interpretation and some analytical difficulty. Any way it is giving rise to very interesting applications in a lot of fields [30] – see [33] for other references – as Physics [7,11,12,27,28], Biology [59], Biomedical Engineering [19,29,34], Financial market [30], Signal Processing [33] to refer only a few.

Besides the electromagnetism that we will consider next, Fractional Calculus is an essential tool for long memory and long range processes. This nonlocality in time and space can be found in a lot of phenomena, as the diffusion [34,35,38–40]. The anomalous porous media [53] and in general the fractional spaces [6] have been also an interesting subject of study. The application of Fractional Calculus to modelling mechanical systems has been growing also, from the 1-D case [66] to the n-D [61–63]. This has been done through fractional generalisation of Hamiltonian and gradient dynamical systems [11,61–63]. The viscoelasticity is another subject of active research [36]. The general case of field theories has been considered by Herrmann [17,18].

[1] Also with INESC-ID, Lisbon.
* Corresponding author.
E-mail addresses: mdo@fct.unl.pt (M.D. Ortigueira), mrivero@ullmat.es (M. Rivero), jtrujill@ullmat.es (J.J. Trujillo).

http://dx.doi.org/10.1016/j.cnsns.2014.09.004
1007-5704/© 2014 Elsevier B.V. All rights reserved.

1.2. On the fractional Electromagnetism

We can state the beginning of the fractional Electromagnetism on the famous Curie Law (1889) not understandable in the context of classic theory [68]. This is an empirical law stating that the current through a capacitor is given by the following power law $i(t) = h.V_0 t^{\alpha},\ t > 0$, where h and $0 < \alpha < 1$ are constants, V_0 is the DC voltage applied at $t = 0$. A second example is the relaxation phenomena in dielectrics; it was shown to be better described by a fractional model (Cole–Cole [8,9]). Recently a suitable modification to their formula allowed the modelling of the complex permittivity of several dielectric materials [67]. The Constant Phase Elements and in particular the Warburg impedance constitute a group of devices that are represented in amplitude Bode plot by a straight line with slope less than 20 dB per decade and a constant phase less than $\pi/2$ [5]. For instance the Electrochemical Capacitors (ECs) [10,37], also known as double layer capacitors, supercapacitors or ultracapacitors, accumulate the electric energy like the traditional capacitors, but they: (a) have a large capacitance (of the order of thousand Farad); (b) have very small active resistance (some milli Ohm or hundreds micro Ohm); (c) are able to perform several hundred thousand charge and discharge cycles; (d) have the outstanding characteristic of high power delivery; (e) have long useful life, and (f) supply a power density considerably higher than that of batteries. These characteristics make them suitable for a variety of current applications where short, high-power pulses are required such as hybrid electric vehicle, power electronics and telecommunications. The main difference between ECs and conventional capacitors lies in the energy storage. The ECs use the Electric Double Layer for accumulating of the electric energy. Each consists of two electrodes and the space between them is filled with an electrolyte. At the middle there is a separator. The correct model is fractional [23,37].

Two other interesting applications were presented recently: a Magnetoresistive Current Sensor [3] and a CMOS-Based Terahertz Integrated Circuit that was modelled as a Causal Fractional-Order RLGC Transmission Line Model [57]. Also a fractional coil was studied in [58]. Other fractional electric behaviours can be found in Nature as in fruits [22] and biomaterials [34]. The spatial dispersion in Electrodynamics as a fractional phenomenon was studied in [64]. Meanwhile the study of fractional circuits has been done in several papers [24,31,2,15].

The description of these examples had in mind motivate the need for a development of fractional Maxwell equations. Several attempts into this goal have been done. The first can be considered the introduction of a fractional curl by Engheta [14]. Since then several proposals have been done [4,13,20,41,42,56,60]. However we can not say that they are satisfactory enough to be accepted as true generalisations of the classic equations. We will try to overcome this problem.

1.3. New results

The generalisation of the well-known vectorial operators: gradient, divergence, curl, and laplacian is a urgent need for enlarging their application field. Some attempts have been made, for example in [12,38,60], but we can say that much work remains to be done. Without intending to do analysis of such proposals, we are going to introduce a new approach. We start by the space version of the fractional forward and backward Grünwald–Letnikov derivatives as presented in [47]. From them we define two fractional gradients and from them the other operators: divergences, curls and laplacians. These operators are backward compatible in the sense that they recover the classic when the order is 1.

An important aspect of our formulation concerns the use of different orders for different directions. This non-local tools that, unlikely to the classic, look more suitable for dealing with non homogeneous and anisotropic media. Although we use constant derivative orders there is nothing impeding us to make them variable. This would enlarge the application range to very complex media.

With those operators we reformulate the classic Helmholtz decomposition theorem for fractional space and time, which maybe is the main result. From it we deduce two sets of fractional partial differential equations for two intensity fields in a Maxwell fashion that lead to the fractional Maxwell equations generalising the classic equations base of the classical Electromagnetism theory.

1.4. Remarks

- We will assume that we are working on $\mathbf{R}^3$ and the set $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ constitutes its standard orthonormal base.
- Each vector has the representation

$$\mathbf{v} = v_x\mathbf{e}_1 + v_y\mathbf{e}_2 + v_z\mathbf{e}_3$$

- A scalar function f is defined on $\mathbf{R}^3$ and represented by $f(x, y, z)$.
- A vectorial function will be represented by

$$\mathbf{f}(x, y, z) = f_x(x, y, z)\mathbf{e}_1 + f_y(x, y, z)\mathbf{e}_2 + f_z(x, y, z)\mathbf{e}_3$$

  In particular, we put $\mathbf{r} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$
- When necessary we will represent by $\bar{\alpha}$ the triple $(\alpha_1, \alpha_2, \alpha_3)$ and by $\bar{\omega}$ the triple $(\omega_1, \omega_2, \omega_3)$.
- We will use the two-sided Laplace transform (LT) of $f(x, y, z)$ defined on $\mathbf{R}^3$ and given by:

Download English Version:

https://daneshyari.com/en/article/7155704

Download Persian Version:

https://daneshyari.com/article/7155704

Daneshyari.com