Commun Nonlinear Sci Numer Simulat 22 (2015) 1068–1083

Contents lists available at ScienceDirect

Commun Nonlinear Sci Numer Simulat

journal homepage: www.elsevier.com/locate/cnsns

Burgers equation with time-dependent coefficients and nonlinear forcing term: Linearization and exact solvability

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ARTICLE INFO

Article history:
Received 2 February 2014
Received in revised form 27 June 2014
Accepted 24 August 2014
Available online 30 August 2014

Keywords:
Forced Burgers equation
Linearization
Point transformation
Exceptional orthogonal polynomials

ABSTRACT

We construct and discuss a linearization method for solving Burgers equation with time-dependent coefficients and a nonlinear forcing term. Our results are shown to contain and generalize recent findings (Miskinis, 2001; Buyukasik and Pashaev, 2013). As applications of our method we solve several initial- and boundary-value problems for Burgers equation with forcing of sinusoidal, polynomial, as well as X_1-Laguerre exceptional orthogonal polynomial type.

1. Introduction

Burgers equation is one of the most fundamental tools for describing nonlinear diffusion and dissipation phenomena. It was derived from the Navier–Stokes equations by dropping the pressure term and was proposed as a model of turbulence in hydrodynamic motion [4]. Since then Burgers equation has been found applicable to a wide variety of physical models, the most important of which include standing waves and resonance in opto-acoustic systems [21], non-steady-state forced vibrations in acoustic resonators [7], nonlinear standing waves in constant-cross-sectioned resonators [2], 1-D nonlinear dynamics of hydrodynamic-type fields [22]. Further applications concern soil–water flow in layered media [3] [12], the formation and propagation of soliton- and shock waves [23], acoustic streaming [10], population dynamics [17], and many more. For a detailed overview of modern applications and related mathematical methods the reader may refer to [11] or [25] and references therein. Due to the importance of Burgers equation, there is a general interest in particular cases, where solutions can be expressed in closed form. Such solutions can be constructed by several methods, for example through Lie symmetries [14], the Hirota method [18], the Backlund transformation [19], among others. In addition, one of the simplest and most popular schemes to solve Burgers equation is to linearize it by means of the Cole–Hopf transformation [6] [13] to an equation of Schrödinger type, including the heat equation as a particular case. This method of linearization has been used extensively in order to generate closed-form solutions of Burgers equation, in particular for cases where an external force field is included. Newer results on such cases include purely time-dependent [16] and linear forcing, see [8] and references therein. Very recently, the latter setting was extended to Burgers equation for time-dependent coefficients [5]. It turned out that linearizability to the heat equation persists if a certain interrelation between the coefficients in the equation holds. Now, it is well-known that Burgers equation for a nonlinear forcing term can only be linearized to a Schrödinger-type equation for a nonzero potential. Note that the expression "nonlinear" refers to the forcing term as being a nonlinear function in the spatial variable. In some cases, the latter equation can be further simplified by transforming it into its stationary counterpart,

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http://dx.doi.org/10.1016/j.cnsns.2014.08.029
1007-5704/

which is an ordinary differential equation [9]. It is therefore desirable to have a transformation that takes Burgers equation for time-dependent coefficients and nonlinear forcing term into a stationary Schrödinger equation. The construction of such a mapping, the discussion of its properties, and the presentation of applications are precisely the purpose of this note. In Section 2 we will introduce different point transformations for linearizing Burgers equation and relating it to its Schrödinger counterpart. Section 3 is devoted to the construction of the final transformation and the discussion of its properties. Applications of our method that involve different initial-value- and boundary-value problems, are presented in Section 4. One of these applications involves a forcing term expressed through exceptional orthogonal polynomials of X_1-Laguerre type.

2. Point transformations

The method that will be constructed in Section 3 is based on point transformations that convert the time-dependent Burgers equation to Schrödinger form. We particularly focus on the stationary Schrödinger equation, closed-form solutions of which are comparably easy to find.

2.1. Stationary and time-dependent Schrödinger equations

We will now briefly review a point transformation that was first introduced in [9]. The purpose of this transformation is to interrelate a class of time-dependent Schrödinger equations to certain stationary counterparts. We start out by considering the time-dependent Schrödinger equation

$$i\Phi_t(x,t) + \frac{1}{2m}\Phi_{xx}(x,t) - V_1(x,t)\Phi(x,t) = 0, \tag{1}$$

where the indices stand for partial differentiation and m is a constant. Furthermore, V_1 represents the potential, which we assume to have the following form, introducing arbitrary differentiable functions A, B, C and a constant phase φ:

$$V_1(x,t) = \exp\left[\frac{4}{m}\int^t A(t')dt' + \frac{4\varphi}{m}\right]V_0[u(x,t)] + \left[A'(t) - \frac{2}{m}A^2(t)\right]x^2 + \left[B'(t) - \frac{2}{m}A(t)B(t)\right]x + C(t). \tag{2}$$

The function V_0 that appears in this expression is assumed to be differentiable and have an argument u of the following form

$$u(x,t) = \exp\left[\frac{2}{m}\int^t A(t')dt' + \frac{2\varphi}{m}\right]x + \frac{1}{m}\int^t \exp\left[\frac{2}{m}\int^{t'} A(t'')dt'' + \frac{2\varphi}{m}\right]B(t')dt'. \tag{3}$$

Let us now assume that the function Ψ is a solution of the time-dependent Schrödinger equation for the stationary potential V_0, that is,

$$i\Psi_t(x,t) + \frac{1}{2m}\Psi_{xx}(x,t) - V_0(x)\Psi(x,t) = 0. \tag{4}$$

Define further the abbreviation

$$v(t) = \int^t \exp\left[4\int^{t'}\frac{A(t'')}{m}dt''\right]dt',$$

then the function Φ, given by

$$\Phi(x,t) = \exp\left\{-iA(t)x^2 - iB(t)x + \int^t\left[\frac{A(t')}{m} - i\frac{B^2(t')}{2m} - iC(t')\right]dt'\right\}\Psi[u(x,t), v(t)], \tag{5}$$

provides a solution of the initial time-dependent Schrödinger equation (1) for the potential (2). In the particular case that Ψ is a solution to the stationary Schrödinger equation

$$\frac{1}{2m}\Psi''(x) + [E - V_0(x)]\Psi(x) = 0 \tag{6}$$

for an arbitrary constant E, then

$$\Phi(x,t) = \exp\left\{-iA(t)x^2 - iB(t)x - iE\int^t \exp\left[\frac{4}{m}\int^{t'} A(t'')dt''\right]dt' + \int^t\left[\frac{A(t')}{m} - i\frac{B^2(t')}{2m} - iC(t')\right]dt'\right\}\Psi[u(x,t)], \tag{7}$$

solves our initial time-dependent Schrödinger equation (1) for the potential (2). Hence, if in the latter case a solution to the stationary equation (6) is known, then a corresponding solution for the fully time-dependent system (1) and (2) can be constructed by means of (7).