Commun Nonlinear Sci Numer Simulat 22 (2015) 1243–1251

Contents lists available at ScienceDirect

Commun Nonlinear Sci Numer Simulat

journal homepage: www.elsevier.com/locate/cnsns

# Enhanced group classification of Gardner equations with time-dependent coefficients

Olena Vaneeva [a,*], Oksana Kuriksha [b], Christodoulos Sophocleous [c]

[a] Institute of Mathematics of NAS of Ukraine, 3 Tereshchenkivska Str., 01601 Kyiv-4, Ukraine
[b] Petro Mohyla Black Sea State University, 10, 68 Desantnykiv Street, 54003 Mykolaiv, Ukraine
[c] Department of Mathematics and Statistics, University of Cyprus, Nicosia CY 1678, Cyprus

ARTICLE INFO

*Article history:*
Received 5 August 2014
Accepted 18 September 2014
Available online 28 September 2014

*Keywords:*
Group classification
Lie symmetries
Equivalence transformations
Gardner equations
mKdV equations

ABSTRACT

We classify the Lie symmetries of variable coefficient Gardner equations (called also the combined KdV–mKdV equations). In contrast to the particular results presented in Molati and Ramollo (2012) we perform the exhaustive group classification. It is shown that the complete results can be achieved using either the gauging of arbitrary elements of the class by the equivalence transformations or the method of mapping between classes. As by-product of the second approach the complete group classification of a class of variable coefficient mKdV equations with forcing term is derived. Advantages of the use of the generalized extended equivalence group in comparison with the usual one are also discussed.


## 1. Introduction

Lie symmetry analysis proved itself as a powerful and algorithmic tool for studying differential equations (DEs). In spite of its original goal of finding exact solutions for DEs (especially for nonlinear ones) Lie symmetries have been found useful in construction of conservation laws [14], seeking fundamental solutions [5], solving initial and boundary value problems [3], construction of numerical solutions (see, e.g., [22]), study of complicated systems using invariant submodels [17], derivation of physically important models using the requirement of invariance under certain group of transformations (like, e.g., Galilei or Poincaré group) [6], etc.

One of the central problems of group analysis is the *group classification problem* that concerns not a single DE but a class of DEs (DE or a system of DEs that is parameterized by arbitrary elements being constants and/or functions). The solution of the problem implies finding the Lie symmetry group admitted by any DE from a given class and deriving all inequivalent values of arbitrary elements for which the corresponding DEs possess Lie symmetry extensions.

There is unceasing interest in solving group classification problems for various classes of DEs that are of current or potential interests for applications. Many such classes involve several arbitrary functions (variable coefficients), which often makes their symmetry analysis difficult. To overcome these obstacles a number of useful tools and notions were proposed. These are, in particular, notions of generalized [11] and extended [7] equivalence groups, admissible [20] (form-preserving [10], allowed [25]) transformations, equivalence groupoid [18], normalized class of DEs [20], contractions of equations and corresponding symmetries [8,24]; the method of furcate split [19], the partition of a non-normalized class into normalized subclasses [2,20], the method of mapping between classes [23].

* Corresponding author.
*E-mail addresses:* vaneeva@imath.kiev.ua (O. Vaneeva), oksana.kuriksha@gmail.com (O. Kuriksha), christod@ucy.ac.cy (C. Sophocleous).

http://dx.doi.org/10.1016/j.cnsns.2014.09.016
1007-5704/

Nevertheless, there is still a number of works where such tools are neglected and only particular results are derived instead of complete classifications. One of such works is the recent classification of the variable coefficient Gardner equations

$$u_t + k(t)uu_x + f(t)u^2u_x + g(t)u_{xxx} = 0, \quad fg \neq 0, \tag{1}$$

presented in [13]. Here $k,\ f$, and $g$ are smooth functions of the variable $t$.

In the present paper we achieve the exhaustive classification using the groups of equivalence transformations of class (1) that are found in Section 2. We show that even the use of usual equivalence group allows one to get the complete result. At the same time utilizing wider generalized extended equivalence group provides more simplification and therefore is preferable. This is illustrated in the process of finding Lie symmetries of equations from class (1) in Section 3.1. We check the obtained results using the alternative method of mapping between classes in Section 3.2. As by-product of the latter approach the exhaustive Lie symmetry classification of the related class of variable coefficient mKdV equations with forcing term is derived. A discussion on optimal choice of the method and a brief comparison of the obtained results with those presented in [13] are given in the conclusion.

## 2. Equivalence transformations

Firstly, we search for nondegenerate point transformations, that preserve the differential structure of the class (1) and change only its arbitrary elements. They are called *equivalence transformations* and form a group. There are several kinds of equivalence groups. The *usual equivalence group*, used by Ovsiannikov for solving group classification problems since late 50's, consists of the nondegenerate point transformations of the independent and dependent variables and of the arbitrary elements of the class, where transformations for independent and dependent variables do not involve arbitrary elements of class [16]. In 1994 Meleshko suggested to consider the *generalized equivalence group*, where transformations of variables of given DEs explicitly depend on arbitrary elements [11,12]. The attribute *extended* for equivalence groups was proposed to distinguish those equivalence groups whose transformations include nonlocalities with respect to arbitrary elements (e.g., if new arbitrary elements are expressed via integrals of old ones) [7].

Given a class of DEs, if we consider the set of triples each of which consists of two fixed equations from class and a point transformation linking them (such triples are called *admissible transformations* and the entire set of them is called *equivalence groupoid* [18]), then equivalence transformations generate a subset in this set. If the set of admissible transformations is generated by the equivalence group of a class, then this class is called *normalized* [20]. The normalization property has appeared to be rather important in group analysis. Thus, algebraic method of group classification guarantees the complete result for normalized classes only [2,20]. It was shown also that a reasonable way of solving group classification problems in classes that are not normalized is the partition of such classes into normalized subclasses [20,24].

Generators of one-parameter subgroups of the equivalence group can be found by the Lie infinitesimal method, whereas the direct method [9,10] allows one to find the entire equivalence group including even discrete equivalence transformations and therefore this technique is preferable. A very useful feature of normalized classes is that the equivalence groups for their subclasses, singled out by setting certain restrictions on arbitrary elements, are subgroups of the equivalence group of the entire class. We will use this property to derive the equivalence group of the class (1).

It was proven in [21] that the more general class of mKdV-like equations

$$u_t + f(t)u^2u_x + g(t)u_{xxx} + h(t)u + (p(t) + q(t)x)u_x + k(t)uu_x + l(t) = 0, \tag{2}$$

where all the parameters are arbitrary smooth functions of $t$, $fg \neq 0$, is normalized in the usual sense. In other words, all point transformations that connect equations from this class are induced by transformations from its usual equivalence group. This group consists of the transformations

$$\tilde t = \alpha(t), \quad \tilde x = \beta(t)x + \gamma(t), \quad \tilde u = \theta(t)u + \psi(t),$$

where $\alpha,\ \beta,\ \gamma,\ \theta$ and $\psi$ run through the set of smooth functions of $t$ and $\alpha_t\beta\theta \neq 0$. The arbitrary elements of (2) are transformed by the formulae [21]:

$$\tilde f = \frac{\beta}{\alpha_t\theta^2}f, \quad \tilde g = \frac{\beta^3}{\alpha_t}g, \quad \tilde k = \frac{\beta}{\alpha_t\theta}\left(k - 2\frac{\psi}{\theta}f\right), \quad \tilde l = \frac{1}{\alpha_t}\left(\theta l - \psi h - \psi_t + \psi\frac{\theta_t}{\theta}\right),$$
$$\tilde h = \frac{1}{\alpha_t}\left(h - \frac{\theta_t}{\theta}\right), \quad \tilde p = \frac{1}{\alpha_t}\left(\beta p - \gamma q + \beta\frac{\psi^2}{\theta^2}f - \beta\frac{\psi}{\theta}k + \gamma_t - \gamma\frac{\beta_t}{\beta}\right), \quad \tilde q = \frac{1}{\alpha_t}\left(q + \frac{\beta_t}{\beta}\right).$$

As class (2) is normalized we are able to derive all admissible transformations of class (1) simply by setting $\tilde l = l = \tilde h = h = \tilde p = p = \tilde q = q = 0$ in the latter formulas. Note that for classes that are not normalized this may lead to incomplete results. As a result we obtain the equations $\beta_t = \theta_t = \psi_t = 0$ and $\beta\psi(\psi f - \theta k) + \gamma_t\theta^2 = 0$. Their solution is $\beta = \delta_1,\ \theta = \delta_2,\ \psi = \delta_3$, and $\gamma = \delta_1\delta_3\delta_2^{-2}\int(\delta_2 k - \delta_3 f)\mathrm{d}t + \delta_4$, where $\delta_i,\ i = 1,\dots,4$, are arbitrary constants with $\delta_1\delta_2 \neq 0$. There is no additional constraint for the function $\alpha$, therefore, it is an arbitrary smooth function with $\alpha_t \neq 0$. The following two assertions are true.