Computers and Fluids 000 (2017) 1–14

Contents lists available at ScienceDirect

Computers and Fluids

journal homepage: www.elsevier.com/locate/compfluid

A high-order nonconservative approach for hyperbolic equations in fluid dynamics

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ARTICLE INFO

Article history:
Received 27 February 2017
Revised 7 June 2017
Accepted 7 August 2017
Available online xxx

Keywords:
Nonconservative formulation
Residual distribution
Conservation
Fluid dynamics

ABSTRACT

It is well known, thanks to Lax–Wendroff theorem, that the local conservation of a numerical scheme for a conservative hyperbolic system is a simple and systematic way to guarantee that, if stable, a scheme will provide a sequence of solutions that will converge to a weak solution of the continuous problem. In [1], it is shown that a nonconservative scheme will not provide a good solution. The question of using, nevertheless, a nonconservative formulation of the system and getting the correct solution has been a long-standing debate. In this paper, we show how to get a relevant weak solution from a pressure-based formulation of the Euler equations of fluid mechanics. This is useful when dealing with nonlinear equations of state because it is easier to compute the internal energy from the pressure than the opposite. This makes it possible to get oscillation-free solutions, contrarily to classical conservative methods. An extension to multiphase flows is also discussed, as well as a multidimensional extension.

1. Introduction

According to the Lax–Wendroff theorem, it is well known that, when considering the numerical approximation of a system of hyperbolic PDEs written in conservative form, the numerical scheme must be written in conservation form, too. It is also known that, for a sequence of meshes with characteristic sizes tending to zero, if a sequence of solutions remains bounded and if its subsequence converges in some norm in L^p, $p \geq 1$, then the limit solution is the weak solution of the original PDE. Moreover, if the scheme satisfies a discrete entropy inequality, then the limit solution will automatically satisfy an entropy inequality. If conservation is lost, then there is no hope to get any meaningful solution, see [1] for the analysis.

However, for engineering purposes, the conservative formulation of the behavior of a mechanical system is not necessarily the best one. Consider for example the Euler equations of fluid dynamics. The system of PDEs is

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho\\ \rho\mathbf{u}\\ E\end{pmatrix} + \operatorname{div}\begin{pmatrix}\rho\mathbf{u}\\ \rho\mathbf{u}\otimes\mathbf{u}+p\,\mathrm{Id}\\ (E+p)\mathbf{u}\end{pmatrix} = 0, \tag{1}$$

supplemented by initial and boundary conditions. As usual, ρ stands for the density, $\mathbf{u}$ for the velocity, and the total energy is

$$E = e + \frac{1}{2}\rho\mathbf{u}^2.$$

The pressure p is related to these variables via an equation of state (EOS):

$$p = p(\rho, e) = p\left(\rho, E - \frac{1}{2}\rho\mathbf{u}^2\right). \tag{2}$$

The system (1) is hyperbolic if $\kappa := \frac{\partial p}{\partial e} > 0$ since the speed of sound c is defined by

$$c^2 = \kappa h, \quad h = \frac{e+p}{\rho}.$$

However, for engineering purposes, the relevant variables are not the conserved ones but rather the primitive ones, namely density, velocity and internal energy or pressure. When the solution is smooth, system (1) can be equivalently written as:

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho\\ \rho\mathbf{u}\\ e\end{pmatrix} + \begin{pmatrix}\operatorname{div}\rho\mathbf{u}\\ \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}+p\,\mathrm{Id})\\ \mathbf{u}\cdot\nabla e + (e+p)\operatorname{div}\mathbf{u}\end{pmatrix} = 0 \tag{3}$$

or

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho\\ \rho\mathbf{u}\\ p\end{pmatrix} + \begin{pmatrix}\operatorname{div}\rho\mathbf{u}\\ \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}+p\,\mathrm{Id})\\ \mathbf{u}\cdot\nabla p + \rho c^2\operatorname{div}\mathbf{u}\end{pmatrix} = 0. \tag{4}$$

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http://dx.doi.org/10.1016/j.compfluid.2017.08.019
0045-7930/

These equations are valid for smooth flows and cannot be considered for discontinuities. Nevertheless, there have been several attempts to solve the Euler equations in formulation (3) or (4), including solutions with shocks. One example of such method is Karni's hybrid scheme [2,3], where formulation (4) is used along slip lines only thanks to a switch in the scheme. In any case, this method violates strict conservation.

This has been a long ongoing debate on how to, nevertheless, use formulations (3) or (4) for the numerical approximation of Euler equations valid for all kinds of flows, e.g. with complex equations of state. In the case of nonlinear equations of state, i.e. when the pressure explicitly depends on the density, the pressure obtained by the numerical scheme cannot be uniform across contact discontinuities. The reason for that behaviour is that, on one hand, the density is evaluated from the mass conservation, and, on the other hand, one evaluates the pressure via energy and density. If, in addition, we want the pressure to be constant across contact discontinuities, this puts a constraint that is in general not compatible with the updated densities, momentum and total energy, see [4] for a short discussion.

Up to our knowledge, in the Eulerian framework, there exists only one approach to this problem which is described in [5]. In this paper, we propose a simpler and more general framework for dealing with nonconservative formulations. We solve equations (3) or (4) in a way which is compatible with local conservation and the continuity of pressure and velocity across contact discontinuities. To achieve this, we rely on a finite volume formulation that uses residuals instead of fluxes. In a flux formulation, the unknowns are approximations of the average values of the conserved variables, and they are balanced by the sum of normal fluxes across the boundary of the control volume. This assumes that the control volume has a polygonal shape. In general, these control volumes are interpreted as cells of a dual mesh which is made of simplices. In the residual formulation, one starts by a mesh whose elements are simplices, and interprets the unknowns as approximations of the point values of the conserved variables. These unknowns, for any given degree of freedom, are then updated by a sum of the local residuals over all the elements that share this degree of freedom. Given any element K, the local conservation is recovered by requiring that the sum of the local residuals for that element is the normal flux over the boundary of K of some consistent approximation of the flux. It is easy to show that any flux formulation leads to a residual form, and the opposite is also true. However, the fluxes that are computed depend not only on the solution on both sides of the face of the control volume, see [6] for details.

The format of this paper is the following. In Section 2, we recall how one can get a residual distribution formulation for the system (3) that is equivalent to a flux formulation of (1). This enables us to get a relation on the increment of the energy that can be generalised for the residual formulation. In Section 3, we show how to use this principle, first on energy-based formulation of the Euler equations (3) and then on the pressure-based formulation (4) for several kinds of equations of state. In Section 4, this is further generalised to multiphase flows where the phases may have very complex equations of state. Finally, we give some concluding remarks in Section 5.

2. From conservative to nonconservative formulation

2.1. A residual formulation of a finite volume scheme

The main advantage of the residual formulation can be understood from the one dimensional setting. Consider the problem:

$$\frac{\partial U}{\partial t}+\frac{\partial f(U)}{\partial x}=0.$$

With standard notations, a generic finite volume writes:

$$U_j^{n+1}=U_j^n-\lambda\big(\hat{f}_{j+1/2}-\hat{f}_{j-1/2}\big)$$

where $\lambda=\Delta t/\Delta x$ and $\hat{f}_{j+1/2}$ is the flux between the states U_j^n and U_{j+1}^n. High order accuracy in space amounts to tune the arguments of the flux, high order in time can be reached via a SSP preserving scheme. To fix the conservation problem one must fix the scheme to recover a flux form, i.e to work directly with the fluxes. This is not easy from the algebraic point of view.

It is known, see for example [7] that any finite volume scheme can be rewritten in terms of a distribution of the residual. Consider for example, and for simplicity, the one dimensional case, its generalisation to any kind of control volume is straightforward, see again [7].

The residual formulation writes (in its simplest form) as

$$U_j^{n+1}=U_j^n-\lambda(\Phi_j^{j+1/2}+\Phi_j^{j-1/2}).$$

The conservation is recovered if for any element $[x_j,x_{j+1}]$ one gets

$$\Phi_j^{j+1/2}+\Phi_{j+1}^{j+1/2}=f(U_{j+1})-f(U_j) \tag{5}$$

for any order of accuracy. For example one can go from a flux formulation to a residual formulation by defining:

$$\Phi_j^{j+1/2}=\hat{f}_{j+1/2}-f(U_j) \text{ and } \Phi_{j+1}^{j+1/2}=f(U_{j+1})-\hat{f}_{j+1/2}.$$

The local conservation is a consequence of relation (5). If we start from a nonconservative formulation in residual form, one can check the conservation if one can provide linear transformations of these residual to obtain a form satisfying (5).

2.2. Unsteady residual distribution formulation for the conservative case

Consider a multidimensional hyperbolic system in the form

$$\frac{\partial U}{\partial t}+\text{div }\mathbf{f}(U)=0.$$

Recall the residual distribution approach from [8]. We start by a Runge–Kutta (RK) formulation: knowing U^n, we define:

$$\begin{aligned}
U^{(0)}&=U^n,\\
\frac{U^{(1)}-U^{(0)}}{\Delta t}&+\text{div }\mathbf{f}(U^{(0)})=0,\\
\frac{U^{(2)}-U^{(0)}}{\Delta t}&+\frac{1}{2}\Big(\text{div }\mathbf{f}(U^{(0)})+\text{div }\mathbf{f}(U^{(1)})\Big)=0,\\
U^{n+1}&=U^{(2)}.
\end{aligned}$$

Next for $l=0,1$ we rewrite each sub-step as:

$$\frac{U^{(l+1)}-U^{(0)}}{\Delta t}+\ \text{DIV }\mathcal{F}(U^{(l)},U^{(0)})=0 \tag{6}$$

where

$$\text{DIV }\mathcal{F}(U^{(l)},U^{(0)})=\frac{1}{2}\Big(\text{div }\mathbf{f}(U^{(0)})+\text{div }\mathbf{f}(U^{(l)})\Big) \tag{7}$$

This will provide an approximation that is second order in time. Without loss of accuracy, we can also write it as

$$\text{DIV }\mathcal{F}(U^{(l)},U^{(0)})=\text{div }\mathbf{f}\Big(\frac{U^{(0)}+U^{(l)}}{2}\Big) \tag{8}$$

and the adapted modifications for the RK scheme.

We assume that the computational domain Ω is covered by non-overlapping simplices $\{K_j\}_{j\in\mathcal{J}}$, $\Omega=\cup_{j\in\mathcal{J}}\{K_j\}$. The elements K_j are segments in 1D, triangles/quadrilateral in 2D and tetrahedrons/hexahedrons in 3D. In order to simplify the notations, we

Please cite this article as: R. Abgrall et al., A high-order nonconservative approach for hyperbolic equations in fluid dynamics, Computers and Fluids (2017), http://dx.doi.org/10.1016/j.compfluid.2017.08.019