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Artificial compressibility Godunov fluxes for variable density incompressible flows

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ABSTRACT

In this work we present and compare three Riemann solvers for the artificial compressibility perturbation of the 1D variable density incompressible Euler equations. The goal is to devise an artificial compressibility flux formulation to be used in Finite Volume or discontinuous Galerkin discretizations of the variable density incompressible Navier–Stokes equations. Starting from the constant density case, two Riemann solvers taking into account density jumps at fluid interfaces are first proposed. By enforcing the divergence free constraint in the continuity equation, these approximate Riemann solvers deal with density as a purely advected quantity. Secondly, by retaining the conservative form of the continuity equation, the exact Riemann solver is derived. The variable density solution is fully coupled with velocity and pressure unknowns. The Riemann solvers are compared and analysed in terms of robustness on harsh 1D Riemann problems. The extension to multidimensional problems is described. The effectiveness of the exact Riemann solver is demonstrated in the context of an high-order accurate discontinuous Galerkin discretization of variable density incompressible flow problems. We numerically validate the implementation considering the Kovasznay test case and the Rayleigh–Taylor instability problem.

1. Introduction

The numerical solution of both the constant and the variable density Incompressible Navier–Stokes (INS) equations is a challenging task. On the one hand explicit integration in time is usually not employed due to the algebraic nature of the incompressibility constraint, on the other hand fully coupled velocity-pressure formulations result in systems of Differential Algebraic Equations (DAEs) that are expensive to solve due to the saddle point nature of the problem. Decoupled time integration strategies based on projection methods, see Chorin [7] and Temam [15], and artificial compressibility methods, see Chorin [6], have been widely employed to improve the effectiveness of the solution strategy. A pressure correction scheme for variable density incompressible flows was devised by Guermond and Quartapelle [9] while Pyo and Shen proposed a Gauge–Uzawa method [12].

In the context of discontinuous Galerkin (dG) formulations of incompressible flow problems, the artificial compressibility concept has been employed to recover hyperbolicity at inter-element boundaries and devise a suitable Godunov numerical flux for velocity-pressure coupling, see e.g. [3,4]. As an alternative approach, recently Tavelli and Dumbser [13,14] proposed to use staggered meshes within the dG framework.

The dG method introduced by Bassi et al. [3] in the context of constant density INS equation is here extended in order to deal with variable density INS equations. In [3] the artificial compressibility is introduced only at the interface flux level to obtain a physically meaningful coupling between pressure and velocity. Accordingly, the resulting INS equations discretization is consistent irrespectively of the amount of artificial compressibility introduced. The artificial compressibility flux allows for equal degree velocity-pressure formulations and provides robustness when dealing with convection-dominated flow regimes but it only mitigates the difficulties involved in the solution process. In the context of fully coupled dG formulations of the INS equations efficiency might be pursued with ad hoc preconditioners, for example a recent work by Botti et al. [5] reports promising results by means of agglomeration based *h*-multigrid solution strategies.

In the present work we introduce the exact and two approximate Riemann solvers for the artificial compressibility perturbation of the 1D variable density incompressible Euler equations. In the approximate Riemann solvers a transport equation for the density unknown was added to the constant density artificial compressibility equations devised by Elsworth and Toro [8]. Then we consider two choices for the density in the momentum equation, *i.e.*, either a constant reference density or the variable physical density.

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Table 1
Sets of equations for the three Riemann solvers here considered.

CDRS	SDRS	ERS
$\frac{1}{\rho_0 c^2}\frac{\partial p}{\partial t}+\frac{\partial u}{\partial x}=0$	$\frac{1}{\rho_0 c^2}\frac{\partial p}{\partial t}+\frac{\partial u}{\partial x}=0$	$\frac{1}{\rho_0 c^2}\frac{\partial p}{\partial t}+\frac{\partial u}{\partial x}=0$
$\frac{\partial(\rho_0 u)}{\partial t}+\frac{\partial(\rho_0 u^2+p)}{\partial x}=0$	$\frac{\partial(\rho u)}{\partial t}+\frac{\partial(\rho u^2+p)}{\partial x}=0$	$\frac{\partial(\rho u)}{\partial t}+\frac{\partial(\rho u^2+p)}{\partial x}=0$
$\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}=0$	$\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}=0$	$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}=0$

In the former case velocity and pressure solutions are decoupled from density fluctuations while in the latter velocity and pressure are influenced by density jumps across the contact discontinuity. In the exact Riemann solver the conservative continuity equation is considered in place of the density transport equation. Only in this setting the density solution might differ from the left and right states. Interestingly, the exact Riemann solver admits an explicit solution, therefore it is also the most efficient. The solvers are designed to be applied in the direction normal to element faces and are well suited to be employed in the context of high–order dG discretizations. To this end we provide a multidimensional extension for each of the proposed solvers.

The paper is organized as follows: in Section 2 we present the two approximate and the exact Riemann solvers for 1D problems. In Section 3 we derive the Riemann solvers for x-split 3D problems. In Section 4.1 we consider five benchmark Riemann problems to compare the numerical fluxes provided by exact and approximate Riemann solvers. Next we turn to the dG discretization of the variable density INS equations, where artificial compressibility is introduced only at the flux level. The convergence properties of the dG formulation are assessed on the 2D Kovasznay test case in Section 4.2. Finally, the robustness of the formulation is assessed considering the Rayleigh–Taylor instability problem in Section 4.3.

2. The 1D Riemann problem

The 1D variable density incompressible Euler equations, modified by means of an artificial compressibility term, are

$$\begin{aligned} \frac{1}{\rho_0 c^2}\frac{\partial p}{\partial t}+\frac{\partial u}{\partial x}&=0,\\ \frac{\partial(\rho u)}{\partial t}+\frac{\partial(\rho u^2+p)}{\partial x}&=0,\\ \frac{\partial \rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}&=0, \end{aligned} \tag{1}$$

where p is the pressure, u the velocity and ρ is the density. Here $c \in \mathbb{R}\backslash\{0\}$ is the artificial compressibility coefficient and $\rho_0 = 1$ is a reference density.

We wish to find the solution of the 1D Riemann problem

$$p,u,\rho=\begin{cases} p_L,u_L,\rho_L & x<x_0, t=0\\ p_R,u_R,\rho_R & x>x_0, t=0\end{cases}, \tag{2}$$

for model (1). Subscripts L and R distinguish initial left and right states respectively, where x_0 is the position of the initial discontinuity. The solution consists of four states separated by two acoustic waves, hereafter called "left" and "right" waves, and a contact discontinuity (see Fig. 1). Left and right waves can be either rarefactions or shocks depending on the initial values and across them all the unknowns can change. Instead, in the region between waves, called *star region* ($\star$), pressure and normal velocity are constant and only the density can vary.

In Table 1 we report, for the sake of comparison, the sets of equations for the Riemann solvers considered in this work. The first two solvers (CDRS and SDRS) are exact solvers based on modified sets of equations as compared to (1) and thus they give rise to approximate solutions.

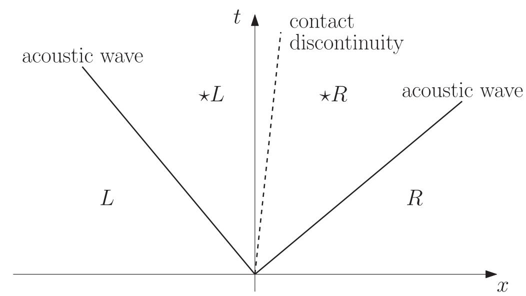

Fig. 1. Structure of the Riemann problem solution in the $x-t$ plane.

2.1. Constant density approximate Riemann solver (CDRS)

The CDRS is the Riemann solver proposed by Elsworth and Toro [8] for constant density flows and based on the following hyperbolic set of equations

$$\begin{aligned} \frac{1}{\rho_0 c^2}\frac{\partial p}{\partial t}+\frac{\partial u}{\partial x}&=0,\\ \frac{\partial(\rho_0 u)}{\partial t}+\frac{\partial(\rho_0 u^2+p)}{\partial x}&=0. \end{aligned} \tag{3}$$

Here, to account for density variations, we simply augment the original set of equations with the density transport equation

$$\frac{D\rho}{Dt}=\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}=0, \tag{4}$$

which allows to consider the density as a purely advected property, *i.e.*, a property that can change only across the contact discontinuity

$$\rho_{\star L}=\rho_L, \qquad \text{and} \qquad \rho_{\star R}=\rho_R. \tag{5}$$

Note that, according to Eq. (3), a reference density $\rho_0 = 1$ is employed in the momentum equation. As a consequence pressure and velocity solutions inside the star region are decoupled from the density fluctuations and can be obtained by means of the solver of Elsworth and Toro [8].

2.2. Switched density approximate Riemann solver (SDRS)

As for the CDRS the SDRS enforces the divergence free constraint inside the continuity equation of the system (1). Accordingly, the density is a purely advected property, and its solution inside the star region reads

$$\rho_{\star L}=\rho_L, \qquad \text{and} \qquad \rho_{\star R}=\rho_R. \tag{6}$$

Since the density can vary only across the contact discontinuity, the first two equations of (1) can be rewritten as

$$\begin{aligned} \frac{1}{\rho_0 c^2}\frac{\partial p}{\partial t}+\frac{\partial u}{\partial x}&=0,\\ \frac{\partial \hat\rho u}{\partial t}+\frac{\partial(\hat\rho u^2+p)}{\partial x}&=0, \end{aligned} \tag{7}$$

where the constant density $\hat\rho$ takes the left/right value on the left/right of the contact discontinuity. As a consequence, even if the