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A numerical technique for applying time splitting methods in shallow water equations

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ABSTRACT

In this paper we analyze the use of time splitting techniques for solving shallow water equations. We discuss some properties that these schemes should satisfy so that interactions between the source term and the shock waves are controlled. This work shows that these schemes must be well balanced in the meaning expressed by Greenberg and Leroux [7]. More specifically, we analyze in what cases it is enough to verify an Approximate C-property and in which cases it is required to verify an Exact C-property (see [1, 2]). We also discuss this technique in two dimensions and include some numerical tests in order to justify our argument.

1. Introduction

Time splitting schemes on balance laws (conservation laws with source terms) is a useful technique for separately treating the homogeneous and the general equation of a system of differential equations with source term. A prototype, in two-dimensional space and under certain regularity hypotheses, is given by the following system of partial differential equations

$$\begin{cases} W(x,y,t)_t + F_1(W(x,y,t))_x + F_2(W(x,y,t))_y \\ \qquad\qquad = G(x,y,W(x,y,t)), \\ W(x,y,0) = W_0(x,y), \quad (x,y,t) \in \mathbb{R}^2 \times \mathbb{R}^+. \end{cases} \tag{1}$$

where $W : \mathbb{R}^2 \times \mathbb{R}^+ \to \mathbb{R}^m$ is the vector of conserved variables, $F_1 : \mathbb{R}^m \to \mathbb{R}^m$ and $F_2 : \mathbb{R}^m \to \mathbb{R}^m$ are the vectors of fluxes and $G : \mathbb{R}^{m+2} \to \mathbb{R}^m$ is the source term (usually $m \geq 2$).

In this paper, we clarify a controversy related on using or not time splitting techniques on hyperbolic equations involving solutions with shock waves and in particular on shallow water equations.

In order to solve the system (1), a time splitting scheme consists of solving consecutively the homogeneous equation

$$W(x,y,t)_t + F_1(W(x,y,t))_x + F_2(W(x,y,t))_y = 0, \tag{2}$$

and the ordinary differential equation

$$W(x,y,t)_t = G(x,y,W(x,y,t)). \tag{3}$$

LeVeque notices in [13,14] that such schemes can easily fail by the presence of shock waves in solving the system (2). These shock waves involve large changes in the solution which can not be captured in solving (3).

Ma, Sun and Yin in [16] use a time integrating scheme with two-step predictor-corrector sequence quite successfully. Striba uses splitting techniques for the term that represent the Coriolis acceleration [18] in meteorology models and Wicker and Skamarock [21] use those techniques for integrating elastic equations.

Bermúdez and Vázquez (see [1,2]) introduce the concept of *exact C-property* and *approximate C-property* in order to identify numerical schemes with an acceptable level of accuracy in the resolution of shallow water equations (well-balanced scheme). In addition, Greenberg and Leroux [7] propose a numerical scheme that preserves a balance between the source terms and internal forces due to the presence of shock waves.

Another point of view is provided by Lubich [15] who gives an error analysis of Strang-type splitting integrators for nonlinear Schrödinger equations.

Holdahl, Holden and Lie [8] use an adaptive grid refinement and a shock tracking technique to construct a front-tracking method for hyperbolic conservation laws. They combine the operator splitting to study shallow water equations. Holden, Karlsen, Risebro and Tao [10] show that the Godunov and Strang splitting methods converge with the expected rates if the initial data are sufficiently regular. The reader can find a deep study of splitting methods for partial differential equations in [9], where some analysis of conservation and balance laws are included.

We analyze the above controversy with the support of the ideas presented in [1,2,7]. From these studies, it follows that the numerical schemes used in solving (2) and (3) cannot be arbitrary, even though they have a high degree of accuracy. These must be

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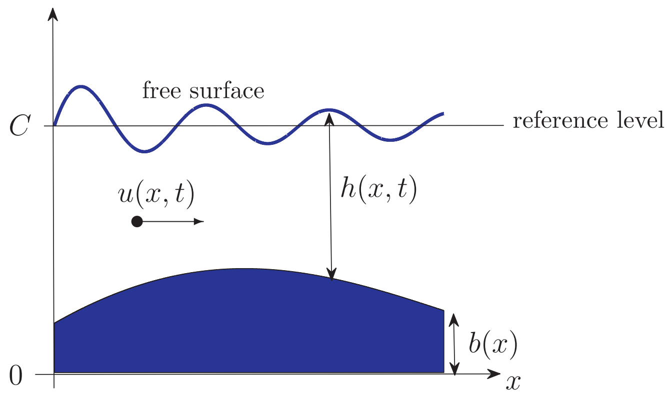

Fig. 1. Shallow water variables.

balanced so that interactions between the source term in (3) and the shock waves in (2) are controlled. In this framework, we will analyze conditions to be satisfied by splitting schemes in order to avoid spurious oscillations which are created in this type of equations. More specifically, we will explain when it is enough to impose an *approximate C-property* and when we need to impose an *exact C-property*.

2. The one dimensional shallow water equations

In this section, we only reproduce the most significant aspects of the one-dimensional case, since it has already been well studied in [17] and the interested reader can easily access it.

2.1. Governing equations

We consider in Eq. (1) the following functions:

$$W = \begin{pmatrix} h \\ q \end{pmatrix},\ F(W) = \begin{pmatrix} q \\ \dfrac{q^2}{h} + \dfrac{1}{2} g h^2 \end{pmatrix},\ G(x, W) = \begin{pmatrix} 0 \\ -ghb'(x) \end{pmatrix}. \tag{4}$$

The unknowns of the problem are: the water height is h and the flow per unit length is $q = hu$. Here u is the average vertical speed in the direction of the axis x (see Fig. 1). F is the flux of conservative variables and $g = 9.81\ \mathrm{ms}^{-2}$ is the acceleration due to gravity. The source term G models the bottom variation given by the function $b(x)$.

Let us consider numerical solvers based upon the decomposition $F(W)_x = A(W)\, W_x$, where

$$A(W) = \begin{pmatrix} 0 & 1 \\ -\dfrac{q^2}{h^2} + gh & 2\dfrac{q}{h} \end{pmatrix} \text{ is the Jacobian matrix of } F(W). \tag{5}$$

So that, system (1)–(4) is hyperbolic ($h > 0$), then $A = X \Lambda X^{-1}$, where

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},\ X = \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix},\ X^{-1} = \frac{1}{\lambda_2 - \lambda_1} \begin{pmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{pmatrix}, \tag{6}$$

where $\lambda_1 = \dfrac{q}{h} + \sqrt{gh}$ and $\lambda_2 = \dfrac{q}{h} - \sqrt{gh}$.

Bermúdez and Vázquez [2] characterize the good behavior of the numerical scheme in the manner in which the scheme approximates a steady solution representing the state of water at rest. They introduce the stationary problem (*Problem SP*) given by $q(x, t) = 0$ and $h(x, t) = C - b(x)$ an independent function of t. Now, they define the following conservation properties:

Exact C-property. We say that a scheme satisfies the exact C-Property if it is exact when applied to the Problem SP.

Approximate C-property. We say that a scheme satisfies the approximate C-Property if it is accurate to the order $\Theta(\Delta x^2)$ when applied to Problem SP.

It is well known that if a numerical scheme does not satisfy any of these conservation properties, then the propagation of spurious oscillations is also present in non stationary problems.

2.2. Central numerical schemes

In order to admit discontinuous solutions (see [19]), we consider the following integral formulation:

$$\int (W dx - F(W) dt) = 0. \tag{7}$$

Numerical schemes generally use (7) in order to approximate (2). To do that, it is introduced a control volume in the space (x, t) of dimensions $\Delta x \times \Delta t$. Next, the integral (7) is evaluated in this control volume

$$\int_{x_{j-1/2}}^{x_{j+1/2}} \left(W(x, t^{n+1}) - W(x, t^n) \right) dx + \int_{t^n}^{t^{n+1}} \Big(F(W(x_{j+1/2}, t)) \\ - F(W(x_{j-1/2}, t)) \Big) dt = 0.$$

Dividing by Δx we obtain

$$\frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} W(x, t^{n+1}) dx = \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} W(x, t^n) dx - \\ - \frac{\Delta t}{\Delta x} \left[\frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} F(W(x_{j+1/2}, t)) dt - \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} F(W(x_{j-1/2}, t)) dt \right],$$

where we have also introduced division by Δt in order to consider integral averages in time of the flux.

Thus, we deduce the conservation formula

$$\overline{W}_j^{n+1} = \overline{W}_j^n - \frac{\Delta t}{\Delta x} [F_{j+1/2} - F_{j-1/2}], \tag{8}$$

where $\overline{W}_j^n$ is the cell average defined as

$$\overline{W}_j^n = \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} W(x, t^n) dx$$

at time $t = t^n$ inside the interval $I_j = [x_{j-1/2}, x_{j+1/2}]$ whose length is $\Delta x = x_{j+1/2} - x_{j-1/2}$.

The flux in (8) can be interpreted as the average in time of the physical flux, i.e.,

$$F_{j+1/2} \approx \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} F(W(x_{j+1/2}, t)) dt.$$

Conservative numerical methods for (2) are based in (8), and they are determined by the expression of the numerical flux $F_{j+1/2}$.

In the first step of the splitting procedure, we solve (2) at each time step. To do that, and for the sake of simplicity, we consider the Q-scheme of van Leer (see [20]) which uses a matrix Q satisfying some properties and the numerical fluxes $F_{j-1/2} = \phi(W_{j-1}^n, W_j^n)$, $F_{j+1/2} = \phi(W_j^n, W_{j+1}^n)$, where the numerical flux function ϕ is given by

$$\phi(U, V) = \frac{F(U) + F(V)}{2} - \frac{1}{2} |Q(U, V)| (V - U).$$

A possible choice of the matrix Q can be the Jacobian (5) of the system (4) evaluated at the arithmetic mean, i.e. $Q(U, V) = A\left(\frac{U+V}{2}\right)$. So that,

$$\left| Q(W_{j\pm\frac{1}{2}}^n) \right| = X(W_{j\pm1}^n, W_j^n) \left| \Lambda(W_{j\pm1}^n, W_j^n) \right| X^{-1}(W_{j\pm1}^n, W_j^n), \tag{9}$$

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