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On existence and uniqueness of entropy solutions of weakly coupled hyperbolic systems on evolving surfaces

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ABSTRACT

In this paper we will present an existence and uniqueness result for an entropy solution of weakly coupled systems of conservation laws on moving surfaces without boundary. The evolution of the hypersurface is prescribed. The coupling of the system is realized by a source term not depending on derivatives of the unknown function.

1. Introduction

Weakly coupled systems of conservation laws on moving surfaces appear in different applications like transport on membranes of biological cells, dynamical combustion [17] and transport of different species on moving water waves. Up to now there are not many papers in which applications of transport processes on moving surfaces are considered. Convection diffusion problems for the transport of surfactants on moving interfaces are considered in [12]. The problem of several surfactants with chemical reactions and without diffusion would fit into the context of this paper and will be considered in the future. These interfaces e.g. occur as interfacial manifolds between phases in multidimensional multiphase flow (see [1,6]). In [18] the dynamics of a gravity-driven thin film flow with insoluble surfactants are described by a different coupled system of PDEs. In order to describe more details let us fix some notations.

Assumption 1. Let Γ_0 be a compact, i.e. without boundary, smooth oriented hypersurface in $\mathbb{R}^{n+1}$ and let $\{(U_i,\ \xi_i)|i\in I)\}$ be a parametrization of this surface. Let $T>0$. We assume that there exists a diffeomorphism $\phi(\cdot\,,t)$: $\Gamma_0 \to \Gamma(t)$ with $\phi \in C^\infty(\Gamma_0\times[0,T])$ and $\phi(\cdot,0)=\mathrm{id}_{\Gamma_0}$, which describes the movement of the surface. Thus we can conclude that the assumptions posed on Γ_0 still hold for Γ_t. We define the space time area:

$$G_T := \bigcup_{t\in(0,T)} \Gamma(t)\times\{t\}.$$

Then the parametrization of G_T is given by $\{(\psi_i,\ U_i)|i\in I\}$, with $\psi_i(x,t)=(\phi(\xi_i(x),t),t)\in C^\infty(U_i\times[0,T])$. Let the flux functions $\vec{f}_l=(f_l^1,\dots,f_l^M)$ with $l\in\{1,\dots,n+1\}$ and $f_l^i\in C^2(\overline{G_T}\times\mathbb{R},\mathbb{R})$ for $i=1,\dots,M$ be given and we assume $\nabla_\Gamma\cdot f^i(\cdot,t,s)=0$ for all fixed $t\in\mathbb{R}^+$, $s\in\mathbb{R}$ and $i\in\{1,\dots,M\}$. Furthermore, let the source term $\vec{r}$ be given with $\vec{r}=(r^1,\dots,r^M)$, $r^i\in C^2(\overline{G_T}\times\mathbb{R}^M,\mathbb{R})$ and we assume that for all $i=1,\dots,M$ there exists a constant $L_r^i\ge 0$, such that

$$|r^i(x,t,\vec{u})-r^i(x,t,\vec{v})|\le L_r^i[|u^1-v^1|+\dots+|u^M-v^M|] \qquad (1)$$

holds for all $(x,t)\in\overline{G_T}$ and for all $\vec{u},\vec{v}\in\mathbb{R}^M$. Moreover, the initial value $\vec{u}_0\in L^\infty(G_T)^M$ is given.

Remark 2. For the regularity result of the parabolic problem we need the given function to be in C^k, for a suitable k and the surface to be smooth. The assumption $f_l^i\in C^2(\overline{G_T}\times\mathbb{R},\mathbb{R})$ for example is needed in the proof of the uniform L^∞ estimate for the regularized solution. It could be possible to weaken this assumptions with approximation arguments but this is not part of this paper.

Now we can formulate the main problem, i.e. the weakly coupled systems of conservation laws on a moving surface. For given Γ_0, T, diffeomorphism $\phi(\cdot,\ t)$ with surface velocity v, where $v(\phi(\cdot,t))=\partial_t\phi(\cdot,t)$, flux functions $\vec{f}_k=(f_k^1,\dots,f_k^M)$ with $k=1,\dots,n+1$, source function $\vec{r}=(r^1,\dots,r^M)$ and initial value $\vec{u}_0$, find a function

$\vec{u}: G_T\to\mathbb{R}^M, \qquad \vec{u}=(u^1,\dots,u^M)$, which satisfies

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$$\begin{aligned}
&\dot u^1(x,t)+u^1(x,t)\nabla_\Gamma\cdot\nu(x,t)+\nabla_\Gamma\cdot(f^1(x,t,u^1))=r^1(x,t,\vec u),\\
&\dot u^2(x,t)+u^2(x,t)\nabla_\Gamma\cdot\nu(x,t)+\nabla_\Gamma\cdot(f^2(x,t,u^2))=r^2(x,t,\vec u),\\
&\quad\vdots\qquad\qquad\qquad\vdots\qquad\qquad\qquad\vdots\\
&\dot u^M(x,t)+u^M(x,t)\nabla_\Gamma\cdot\nu(x,t)+\nabla_\Gamma\cdot(f^M(x,t,u^M))=r^M(x,t,\vec u)
\end{aligned}\tag{2}$$

for $(x, t)\in G_T$ and admits the initial values

$$\vec u(x,0)=\vec u_0(x) \text{ for } x\in\Gamma_0. \tag{3}$$

The coupling of the system is only due to the source terms. The i-th equation in (2) on the left-hand side depends only on u^i. For the derivation of this model see for example [8].

Here $\nabla_\Gamma f(x)$ for $x\in\Gamma_t$ is defined as follows. Let $\tilde f$ be a C^1-extension of f in an open neighborhood of Γ. On Γ we define the tangential gradient $\nabla_\Gamma f$ of f by the projection of the gradient on the tangent space, namely

$$\nabla_\Gamma f(x)=\nabla\tilde f(x)-\nabla\tilde f(x)\cdot\nu(x)\nu(x),\qquad x\in\Gamma,$$

where ∇ denotes the gradient in $\mathbb{R}^{n+1}$ and ν the outer unit normal on Γ. The components of the tangential gradient are denoted by $\nabla_\Gamma f=(\underline{D}_1 f,\dots,\underline{D}_{n+1}f)$. With $\dot u$ we denote the material derivative of the function u, where the material derivative of a C^1 function $f(x, t)$ defined on an open neighborhood of $\Gamma(t)\times t$ is given by:

$$\dot f=\frac{\partial f}{\partial t}+\upsilon\cdot\nabla f,$$

where v is the velocity of the surface.

In the following we present a summary of the existing work done in this field to the best of our knowledge. First, there is the Euclidean case where the weakly coupled systems are considered as Cauchy problems on $\mathbb{R}^2$ in Levy [17] and Rohde [28] or later in several space dimensions in Rohde [29]. Here the authors looked at the model introduced by Majda, that can be found in [21]. Natalini and cooperators examined a similar problem in [25], where they stated some comparison results. In the Euclidean case, for example, O.A. Ladyzenskaja and collaborators dealt with quasi linear parabolic systems [15]. We need this for the regularized problem. Holden et al. extended the weakly coupled systems of hyperbolic conservation laws in [11], looking at the problem with a degenerate diffusion flux function on $\mathbb{R}^d$ and proved existence by showing that a numerical scheme converges to the entropy solution. Further work, with interesting applications in radiation hydrodynamics, chemosensitive movement and numerics for convection dominated parabolic systems can be found in [10,26,30]. Scalar conservation laws on surfaces without coupling have been studied in [8] by Dziuk, Kröner and Müller. We make use of some results from the scalar case. This will be labeled at the corresponding positions. In the paper of Lengeler and Müller [16] as well as in the thesis of Müller [23], the authors proposed the scalar problem on Riemannian manifolds with time-dependent metric. They proved for the compactness result a *TV* estimate. In the paper of Dziuk and Elliott [7] the authors derived a proof for the existence of a weak solution of the parabolic scalar problem on moving surfaces, inter alia, which we use in our regularized problem. More related research is done by Amorim, Ben-Artzi, LeFloch and Panov in [2,4,27], just to name a few.

Several numerical results exist for solving weakly coupled systems on the Euclidean space (e.g.[28]). In the paper of Dziuk, Kröner and Müller [8] the finite volume method for solving hyperbolic differential equations was transfered to moving surfaces. These numerical algorithms could be combined to get a solver for our problem. However, this is not part of this paper. The results of this paper are part of the thesis 'Weakly coupled systems of conservation laws on moving surfaces' [13], where the problem (2) and (3) is studied in detail.

This paper is structured as follows. In Section 2 we begin with basic definitions and results on evolving surfaces, state assumptions on the flux functions and define the term of an entropy solution. In Section 3 we consider the regularized problem with diffusive term scaled with viscosity parameter ε. Here we prove the following results: Existence of a weak solution in Section 3, viscosity parameter-dependent $L^\infty(G_T)$-estimate in Section 3.2, regularity of the solution in Section 3.3. and dummyTXdummy- viscosity-independent $L^\infty(G_T)$-estimate in Section 3.4. In Section 4 we show compactness in space and time for the regularized solution. In particular we get a uniform estimate of the tangential gradient in Section 4.1. and dummyTXdummy- an equicontinuity in the mean with respect to time in Section 4.2. With these results we conclude in Section 5 the convergence of a subsequence of regularized solutions in $L^1(G_T)$. In Section 6 we prove that the limit function fulfills the entropy condition and in Section 7 we show uniqueness of this entropy solution.

2. Notations and assumptions

For a real-valued C^2 function f defined in a neighborhood of Γ, we define the Laplace-Beltrami operator by

$$\Delta_\Gamma f(x):=\nabla_\Gamma\cdot\nabla_\Gamma f(x)=\sum_{i=1}^{n+1}\underline{D}_i\underline{D}_i f(x),\qquad x\in\Gamma.$$

Theorem 3. *Let Γ be an oriented C^2-hypersurface with C^1-boundary $\partial\Gamma$, whose intrinsic unit outer normal to Γ is denoted by μ and $f\in C^1(\Gamma)$. Then the formula of integration by parts on Γ is:*

$$\int_\Gamma\nabla_\Gamma f=-\int_\Gamma fH\nu+\int_{\partial\Gamma}f\mu. \tag{4}$$

Here $\nu\in C^1$ is the outer unit normal of Γ and the mean curvature matrix H of Γ is given by

$$H=-\nabla_\Gamma\cdot\nu. \tag{5}$$

Proof. see [9]. □

Remark 4. The material and the tangential derivative of a function $g\in C^2(G_T)$ do not commute. On moving surfaces one has to consider the following formula

$$(\underline{D}_l g)^{\cdot}=\underline{D}_l\dot g-A_{lr}(\nu)\underline{D}_r g$$

with the matrix $A_{lr}(\nu)=\underline{D}_l\nu_r-\nu\cdot\nu_l\underline{D}_r\nu$ with $l,r=1,\dots,n+1$. The proof can be found in [8], p. 207, Lemma 2.6. Furthermore, we have $A(\nu)\nabla_\Gamma g\cdot\nabla_\Gamma g=D(\nu)\nabla_\Gamma g\cdot\nabla_\Gamma g$, where $D(\nu)_{i,j}=\frac{1}{2}(\underline{D}_i\nu_j+\underline{D}_j\nu_i)$ for $i,j=1,\dots,n+1$.

Lemma 5. *Let $\Gamma(t)$ be a C^2-hypersurface and f a function defined on the C^2 hypersurface G_T, such that the appearing quantities exist, then we have the so called Leibniz formula or transport theorem*

$$\frac{d}{dt}\int_{\Gamma_t}f=\int_{\Gamma_t}(\dot f+f\nabla_\Gamma\cdot\upsilon). \tag{6}$$

Proof. [7], page 291. □

Definition 6. *Let M be aC^2-hypersurface in $\mathbb{R}^n$. A function*

$$f\in L^1_{loc}(M)=\left\{f:M\to\mathbb{R}\,|\,f\in L^1(G'),\forall G'\subset\subset M, G' \text{ open}\right\}$$

has a weak derivative $h_i\in L^1_{loc}(M),\ i\in(1,\dots,n)$, if for all $\varphi\in C_0^\infty(M)$ the equation

$$\int_M f\underline{D}_i\varphi dS=-\int_M h_i\varphi dS-\int_M f\varphi H\nu_i$$

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