Computers and Fluids 167 (2018) 166–179

Contents lists available at ScienceDirect

Computers and Fluids

journal homepage: www.elsevier.com/locate/compfluid

Semi-implicit staggered discontinuous Galerkin schemes for axially symmetric viscous compressible flows in elastic tubes

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A R T I C L E I N F O

Article history:
Received 29 July 2017
Accepted 14 February 2018
Available online 23 February 2018

Keywords:
High order semi-implicit discontinuous Galerkin schemes
Staggered grid
Large time steps
Viscous compressible flows
Elastic pipes
Fast transient regime

A B S T R A C T

We propose a novel family of staggered semi-implicit discontinuous Galerkin (DG) finite element schemes for the simulation of axially symmetric, weakly compressible and laminar viscous flows in elastic pipes. The equation of state (EOS) of the fluid is assumed to be barotropic and two different mathematical models derived from the compressible Navier-Stokes equations are considered in this paper.

The first model describes cross-sectionally averaged 1D flows, including steady and frequency-dependent wall friction effects. The novelty of our numerical method compared to standard DG schemes consists in the use of a *staggered mesh*, where the pressure is defined over a primary grid and the velocity field is defined on edge-based staggered dual control volumes. This approach is well known from classical semi-implicit finite difference schemes for the incompressible Navier–Stokes equations, but it is still quite unusual for high order DG schemes. The continuity equation is integrated over the control volumes that belong to the main grid, while the momentum equation is integrated over the elements of the edge-based staggered dual grid. The nonlinear convective terms are discretized explicitly, while the pressure gradient and the mass flux are discretized implicitly. Up to second order of accuracy in time can be achieved with the so-called θ-method. Inserting the discrete momentum equation in the discrete mass conservation equation leads to a mildly nonlinear algebraic system for the degrees of freedom of the pressure. Such mildly nonlinear systems can be very efficiently solved using the Newton algorithm of Brugnano and Casulli. We observe that the linear part of the mildly nonlinear system is symmetric and positive definite.

The second model is derived from the compressible Navier-Stokes equations in cylindrical coordinates. Assuming hydrostatic flow with constant pressure inside each cross section as well as axial symmetry, only the terms in the axial and the radial direction need to be considered. Therefore, we call the second model the $2D_{xr}$ model. Also in this case we use a staggered mesh for pressure and velocity and thus the same philosophy as for the 1D model can be applied to obtain the discrete pressure system. For the $2D_{xr}$ model a staggered DG scheme is also applied for the computation of the viscous stress tensor in the discrete momentum equation. However, in radial direction the resulting linear system for the friction terms is not symmetric and is thus solved using the Thomas algorithm for block three-diagonal systems.

The use of a semi-implicit DG scheme leads to a very mild CFL condition based only on the fluid velocity and not on the sound speed, which makes the method very efficient, in particular in the limit cases when the speed of sound of the fluid tends to infinity (incompressible fluid) and in the rigid case where the wall strain of the pipe tends to zero. In addition, at every Newton step a symmetric positive definite and well conditioned block three-diagonal linear system is solved for the pressure, using a matrix-free conjugate gradient method. Moreover, when the polynomial degree of the basis and test functions is equal to zero the schemes reduce to classical semi-implicit finite volume methods.

While in the $2D_{xr}$ model the viscous effects in radial direction are directly obtained from first principles via the Navier-Stokes equations, the 1D model requires an additional closure relation for the wall friction. For both models we perform several tests in order to validate the numerical methods for steady and unsteady flows of compressible and nearly incompressible fluids in elastic and rigid tubes. We also

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https://doi.org/10.1016/j.compfluid.2018.02.019

provide numerical convergence results in order to show that the developed schemes achieve high order of accuracy in space.

1. Introduction

Pressurized flows in rigid and elastic pipes are investigated by many branches of applied sciences. For example, the simulation of blood circulation in the human cardio-vascular system is one of the most important topics in the field of biomedical engineering, see e.g. [37–40,43,46] and references therein. Other important applications in civil engineering are networks for water supply or ducts in plants for the production of hydro-electric energy. Moreover, pipe flow simulations are crucial in many industrial technologies such as in breaking, injection and cooling systems, but also for oil and gas pipelines. All these examples listed above can become very complex and require a detailed treatment in order to be well understood and to be well designed. To study the fluid mechanics in such systems, in this paper we develop a new high order approach based on the discontinuous Galerkin finite element method combined with an original *staggered mesh* arrangement of pressure and velocity and a semi-implicit time discretization in order to improve the computational efficiency, in particular in the low Mach number regime. The DG method has first been proposed by Reed and Hill in 1973 for the investigation of neutron transport [44] and later Cockburn and Shu extended these schemes to general systems of nonlinear hyperbolic equations in a famous series of papers [20–24].

The DG method has been applied to the Navier-Stokes equations for the first time by Bassi and Rebay in [3] and by Baumann and Oden in [4,5]. Explicit DG methods suffer from a severe stability restriction on the time step: the higher the polynomial degree of the basis and test functions, the smaller the admissible time step. While for hyperbolic problems the time step decreases only with roughly $1/(2N+1)$, where N is the polynomial degree of the approximation, it scales approximately with $1/(2N+1)^2$ for parabolic terms. This problem can be avoided by adopting an implicit discretization of the equations. An efficient semi-implicit DG scheme for the compressible Navier-Stokes equations was developed by Dolejsi and Feistauer [25] and Tumolo et al. [52], while fully implicit DG methods were developed for example by Bassi et al. in [1,2]. All the aforementioned implicit DG schemes are formulated on *collocated* control volumes and so they typically require the solution of large sparse non-linear systems whose associated linear sub problems are characterized by a rather high condition number. In order to improve the efficiency of implicit DG methods, it is possible to derive semi-implicit schemes on *staggered* control volumes, which leads to linear systems with better properties.

In the context of staggered semi-implicit finite volume and finite difference schemes, an interesting family of methods has been developed for free surface flows by Casulli et al. in a series of papers [10–14,16,17]. The time step of these methods is only restricted by the choice of the discretization used for the nonlinear convective terms and the horizontal viscosity. For explicit upwind discretizations of the convective terms, semi-implicit methods are characterized by a mild CFL condition based only on the fluid velocity and not on the velocity of the surface waves. Moreover, the linear systems to be solved are always symmetric and positive definite. All these features make this approach computationally very efficient. In [15], this class of methods has been applied for the first time to axially symmetric flows in systems of compliant tubes. Later, scalar transport has been investigated in [51] and an extension to fully 3D non-hydrostatic pipe flows has been provided in [33]. In [28,34] the compressibility of the fluid was introduced via a barotropic equation of state, including also a simple cavitation model as well as closure relations accounting for unsteady wall friction for highly transient flow regimes. Semi-implicit finite volume schemes for inviscid and viscous compressible fluids with general equation of state have been very recently developed in [27]. The first high order semi-implicit DG scheme on *staggered* Cartesian grids for the shallow water and the incompressible Navier-Stokes equations has been developed in [26,31]. Later, this method was also extended to unstructured meshes [47–50], as well as to space-time adaptive grids (AMR) [32]. In this paper we will use some of the ideas provided in the previous references in order to develop a new family of semi-implicit DG schemes on staggered grids for the simulation of viscous compressible flows in compliant tubes. Our new numerical schemes are applied to two systems of governing PDEs derived from the compressible Navier–Stokes equations: a two-dimensional $2D_{xr}$ model assuming axial symmetry and a constant pressure within each cross section, as well as a simpler cross-sectionally averaged 1D model. The compliance of the tube wall is described in all cases at the aid of simple algebraic elastic ring models like the Laplace law. Following the approach used in [26], the schemes presented in this paper can be seen as the natural extension to higher order of the numerical methods developed in [15,28,34,51]. The rest of this paper is structured as follows. In Sections 2 and 3 we derive the numerical methods for the 1D and for the $2D_{xr}$ model, respectively. In Section 4 some benchmark problems are presented where the new schemes are validated against available reference solutions. Finally, in Section 5 we give some conclusions and a brief outlook to future research. In this paper we also make use of the Einstein summation convention, implying summation over two repeated indices.

2. Staggered semi-implicit DG scheme applied to the 1D model

2.1. Governing equations of the 1D model

The dynamics of fluids in general time-dependent domains is governed by the compressible Navier–Stokes equations. Supposing axially symmetric flow and that the longitudinal scale is much larger than the radial one, it is possible to simplify the system and to introduce the hypothesis of hydrostatic equilibrium, i.e. the pressure is constant in each cross section. Moreover, we assume laminar flow throughout this paper. Under these assumptions one obtains the following one-dimensional, cross-sectionally averaged model that consists of two PDE, namely the continuity equation and the momentum equation, see [28,30,34,35,51]:

$$\frac{\partial}{\partial t}(\rho A) + \frac{\partial}{\partial x}(\rho A U) = 0, \tag{1a}$$

$$\frac{\partial}{\partial t}(\rho A U) + \frac{\partial}{\partial x}(\rho A U^2) + A\frac{\partial p}{\partial x} = -2\pi R\tau_s - 2\pi R\tau_u, \tag{1b}$$

where $t \in \mathbb{R}_0^+$ is the time and x is the longitudinal coordinate. In the above system (1), $\rho = \rho(x,t)$ is the density, $A = A(x,t)$ is the cross sectional area, $U = U(x,t)$ is the velocity averaged over the section, $p = p(x,t)$ is the pressure, $R = R(x,t) = \sqrt{A/\pi}$ is the tube radius, while $\tau_s = \tau_s(x,t)$ and $\tau_u = \tau_u(x,t)$ are the steady and the unsteady wall shear stress, respectively. Since there are six unknowns in the system (1), four closure relations need to be introduced. The first one is the equation of state for barotropic fluids