Computers and Fluids 166 (2018) 139–151

Contents lists available at ScienceDirect

Computers and Fluids

journal homepage: www.elsevier.com/locate/compfluid

Fourth order Galilean invariance for the lattice Boltzmann method

Martin Geier [a,*], Andrea Pasquali [a,b]

[a] TU Braunschweig, Institute for Computational Modeling in Civil Engineering (iRMB), Braunschweig, Germany
[b] FluiDyna GmbH, Edisonstrasse 3, Unterschleissheim 85716, Germany

ARTICLE INFO

Article history:
Received 4 March 2017
Revised 14 August 2017
Accepted 18 January 2018
Available online 31 January 2018

Keywords:
Lattice Boltzmann
Cumulants
Galilean invariance
Fourth order
Crystallographic lattice Boltzmann
Recursive asymptotic analysis

ABSTRACT

Using the structure of a recursive asymptotic analysis we derive conditions on cumulants that guarantee a prescribed order of Galilean invariance for lattice Boltzmann models. We then apply these conditions to three different lattice Boltzmann models and obtain three models with fourth order accurate advection. One of the models uses 27 speeds on a body centered cubic lattice, one uses 33 speeds on an extended Cartesian lattice and one uses 27 speeds plus three finite differences on a Cartesian lattice. All models offer too few degrees of freedom to impose the conditions on the cumulants directly. However, the specific aliasing structure of these lattices permit fourth order accuracy for a model specific optimal reference temperature. Our theoretical derivations are confirmed by measuring the phase lag of traveling vortexes and shear waves.

1. Introduction

For Eulerian (i.e. fixed grid) methods used in computational fluid mechanics Galilean invariance can, in general, only be obtained within a finite order of approximation. In particular the violations of Galilean invariance in the lattice Boltzmann method are widely discussed in literature [1–7]. Within the limits of second order accuracy the problem is in general solved. However, beyond the second order there are several spurious dependencies of the solution on the frame of reference. These include: a dependance of the viscosity on the flow speed, spurious couplings between moments relaxing with different relaxation rates (in the case of the multi-relaxation-time lattice Boltzmann model), and a phase lag in the advection of vortexes. The first two problems have been solved with the introduction of the cumulant lattice Boltzmann method [6] using a transformation to Galilean invariant mutually uncorrelated observable quantities before collision. What has not yet been solved is the phase lag problem in the advection of traveling vortexes in a superimposed velocity field. This error apparently cannot easily be removed within the usual discretization of the lattice Boltzmann model, at least not without introducing more discrete velocities than usually used.

In order to increase the asymptotic accuracy of the lattice Boltzmann method with regard to Galilean invariance to fourth order all velocity moments up to order four have to be sufficiently Galilean invariant. This implies the application of a very large set of discrete velocities which is undesirable for reasons of efficiency. Instead it is desirable to find a specific discretization that reduces the number of required discrete velocities. This is in theory possible through the aliasing structure of a velocity set. Each finite set of velocities has an infinite number of moments of which only a finite number is independent. In some cases it is possible to design a specific finite velocity set in such a way that the dependent moments turn out to be correct within a required order of accuracy. Hence, in such a case it is not necessary to introduce new variables for additional discrete speeds and the computational efficiency is largely enhanced.

In this paper we investigate different possibilities to remove the phase lag Galilean invariance problem of the cumulant lattice Boltzmann method and make the Galilean invariance of the model fourth order accurate. This is done here in three different ways: by using a different arrangement of the discrete speeds than in the original method; by using more speeds; and by using finite differences to repair the original method (hybrid model).

The reminder of the paper is organized as follows: In Section 2 we address the different possibilities to design lattices based on the cubic Bravais lattice structures. Section 3 introduces the recursive asymptotic analysis technique in diffusive scaling (where we assume that the time step scales with the square of the grid spacing) used for deriving equivalent partial differential equations of the lattice Boltzmann method. In Section 4 we give a brief introduction to cumulants. In Section 5 we derive conditions for Galilean invariance based on the results of the previous two sections. Section 6 introduces the hybrid model. Section 7

* Corresponding author
E-mail address: geier@irmb.tu-bs.de (M. Geier).

https://doi.org/10.1016/j.compfluid.2018.01.015

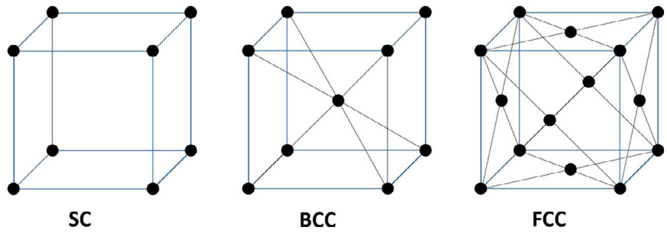

Fig. 1. The three cubic Bravais lattices from left to right: simple cubic (SC), body centered cubic (BCC), and face centered cubic (FCC). These three lattices represent all possible periodic configurations of space filling cubic arrangements of nodes. The simple cubic lattice represents the Cartesian case which is used in most lattice Boltzmann methods. The other two cases have also been used for lattice Boltzmann models: the body centered cubic lattice in Namburi's RD3Q27 method [8] and the face centered cubic lattice in the d'Humières D3Q13 method [10].

discusses some implementation issues of the three models. Section 8 presents the numerical confirmation of our derivations, followed by Section 9 with the conclusions.

2. Crystallographic lattice Boltzmann models

The most commonly applied lattice Boltzmann models use a Cartesian distribution of nodes which, in terms of crystallographic unit cells, corresponds to a simple cubic configuration. Recently, Namburi et al. introduced a lattice Boltzmann discretization based on a body centered cubic unit cell and called this method "crystallographic" lattice Boltzmann [8] arguing that it was inspired by the Bravais lattices used in crystallographic theory [9]. However, we will call Namburi's lattice body centered cubic (BCC) instead of crystallographic due to the fact that the usual simple cubic (SC) discretization is a Bravais lattice too. Yet we stick to the nomenclature of [8] and refer to the velocity distribution of the BCC lattice by RD3Q27 to distinguish it from the Cartesian lattice Boltzmann velocity distribution using 27 speed (D3Q27).

All possible space filling crystallographic lattices can be classified into 14 Bravais lattices of which only three are cubic and hence of interest for approximately isotropic discretizations (see Fig. 1). These are the simple cubic (SC) lattice, the body centered cubic (BCC) lattice and the face centered cubic (FCC) lattice. All three lattices have been used as the basis for lattice Boltzmann models. The popular standard Cartesian lattices with 15, 19 and 27 speeds in three dimensions are SC lattices. Namburi's method uses 27 speeds on a BCC lattice and the so called D3Q13 method uses a FCC lattice [10–12].

According to the theory of Bravais lattices there is nothing beyond this three possibilities, unless unsymmetrical lattices would be considered.

We observed in the past that the standard cumulant lattice Boltzmann method with 27 speeds on a SC lattice lacks Galilean invariance of fourth order only in certain directions [6]. We conjectured that a BCC lattice with the same number of speeds could be more isotropic and should hence be a better starting point for a complete fulfillment of fourth order accuracy of the Galilean invariance. We will therefore investigate the lattice structure proposed by Namburi et al. [8]. In addition, we use a model with 33 speeds on a Cartesian grid to enforce Galilean invariance. We also propose one additional model supplementing the standard D3Q27 lattice with three finite differences to obtain Galilean invariance at fourth order. All lattice used in this study are shown in Fig. 2.

For giving an explicit definition of the used velocity models we introduce the following energy shells:

$$E_0 = \{0, 0, 0\}, \tag{1}$$

$$E_1 = \{0, 0, \pm 1\} \cup \{0, \pm 1, 0\} \cup \{\pm 1, 0, 0\}, \tag{2}$$

$$E_{\sqrt{2}} = \{0, \pm 1, \pm 1\} \cup \{\pm 1, 0, \pm 1\} \cup \{\pm 1, \pm 1, 0\}, \tag{3}$$

$$E_{\sqrt{3}} = \{\pm 1, \pm 1, \pm 1\}, \tag{4}$$

$$E_{\sqrt{3/4}} = \{\pm 1/2, \pm 1/2, \pm 1/2\}, \tag{5}$$

$$E_2 = \{0, 0, \pm 2\} \cup \{0, \pm 2, 0\} \cup \{\pm 2, 0, 0\}. \tag{6}$$

The SC D3Q27 lattice uses the velocity set $\{i, j, k\}_{D3Q27} \in E_0 \cup E_1 \cup E_{\sqrt{2}} \cup E_{\sqrt{3}}$. The BCC lattice uses the set $\{i, j, k\}_{RD3Q27} \in E_0 \cup E_1 \cup E_{\sqrt{2}} \cup E_{\sqrt{3/4}}$ and the D3Q33 lattice the set $\{i, j, k\}_{D3Q33} \in E_0 \cup E_1 \cup E_{\sqrt{2}} \cup E_{\sqrt{3}} \cup E_2$. The D3Q27F3 lattice uses the same velocity set as the D3Q27 lattice plus three simple finite difference stencils.

3. Recursive asymptotic analysis

For the assessment of the convergence order of the lattice Boltzmann method, the rigorous approach is to apply an asymptotic expansion to the lattice Boltzmann equation [13]. The lattice Boltzmann equation for the pre-collision particle velocity distribution function $f_{ijkxyzt}$ can be written as:

$$\begin{aligned} &f^*_{ijk(x+ic\Delta t/2)(y+jc\Delta t/2)(z+kc\Delta t/2)(t+\Delta t/2)} \\ &\quad - f_{ijk(x-ic\Delta t/2)(y-jc\Delta t/2)(z-kc\Delta t/2)(t-\Delta t/2)} = 0. \end{aligned} \tag{7}$$

With the lattice speed $c = \Delta x/\Delta t$ and i, j and k being the quantum numbers for particles moving in x, y and z direction. The range of the quantum numbers depends on the velocity set. The asterisk indicates the post-collision state. In order to eliminate the pre- and

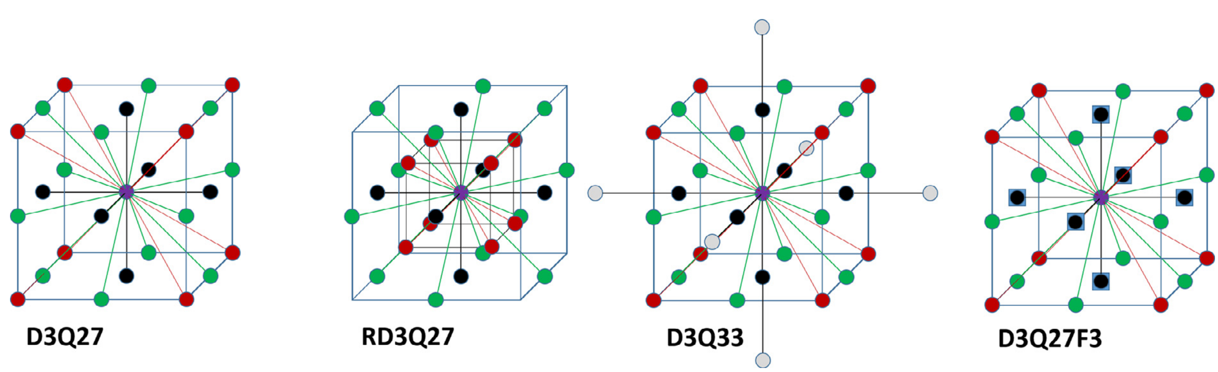

Fig. 2. The four different velocity lattices used in this study. The two lattices on the left have 27 degrees of freedom each. The two lattices on right have 33 degrees of freedom. The D3Q27F3 lattice is a hybrid model that uses 27 distributions together with six pseudo distributions (indicated by squares) used to compute finite differences.