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An improved third-order weighted essentially non-oscillatory scheme achieving optimal order near critical points

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ABSTRACT

In this paper, we present an improved third-order weighted essentially non-oscillatory scheme to recover its convergence order at critical points. This scheme is constructed with the way of Taylor expansion of the smoothness indicators for third-order convergence. The recovery of design-order is verified by several linear wave tests. The performance of the proposed scheme is demonstrated on a variety of one-dimensional problems like the Shu-Osher problem, Titarev-Toro problem and the Interacting blast wave problem, As well as two-dimensional problems such as Rayleigh–Taylor instability, Two-dimensional Riemann problem and Double Mach reflection of a strong shock problem. Numerical results indicate that the present scheme provides better results in comparison with the WENO-JS3, WENO-Z3 and WENO–N3 schemes.

1. Introduction

Weighted essentially nonoscillatory (WENO) schemes are a popular class of numerical schemes for hyperbolic conservation laws. They were first proposed in [10] and then improved by Jiang and Shu [8]. However, In [7], the authors find that this scheme fails to achieve the optimal accuracy at critical points. A mapped function is applied on the original weights to generate the new weights, which satisfy the sufficient condition for optimality of the order, resulting in the fifth-order WENO-M scheme. Inspired by the study in [7], Borges et al. [2] proposed the fifth-order WENO-Z scheme, which were based on available and previously unused information of the classical WENO scheme: a global smoothness indicator of higher order, obtained via a linear combination of the original smoothness indicators. The new scheme can achieve superior results with almost the same computational effort of the classical WENO method. Castro et al. [3] developed a general formula for the higher order smoothness indicators and extended the WENO-Z scheme to all odd orders of accuracy.

Several approaches have been considered for the improvement of the WENO-Z scheme in recent years. One of the effective approaches is through the design of the smoothness indicators. Don and Borges [4] discussed the theoretical analysis on the accuracy of the high-order WENO-Z schemes based on the parameters introduced in the formulation of nonlinear weights. In their study, they have considered the global smoothness indicator as a linear combination of the local smoothness indicator to design a high-order smoothness indicator for the third-order WENO scheme. Ha et al. [6] proposed the WENO–NS scheme using the L1-norm-based smoothness indicators. Yamaleev et al. [19] developed a new third-order Energy Stable Weighted Essentially Non-Oscillatory (ESWENO) finite difference scheme based on new weight functions for scalar and vector hyperbolic equations with piecewise continuous initial conditions. Wu and Zhao [18] proposed an improved third-order WENO-Z scheme (WENO–N3) through constructing a new reference smoothness indicator by a linear combination of the candidate and global smoothness indicator. In order to achieve the desired order of convergence at the critical point, Wu et al. [17] later presented another reference smoothness indicator for third-order WENO-Z scheme by the nonlinear combination of the candidate and global smoothness indicators. Very recently, based on the works in [17,18], Gande et al. [5] have proposed a new reference smoothness indicator by the linear combination of the first derivative information of the local and global stencils to achieve desired order of convergence at critical points, and we denote this scheme as WENO–NF3 scheme in the present paper). It should be mentioned that the common characteristic of improved schemes (WENO-Z and WENO-M schemes) is that they both assign larger weights to less-smooth stencils and while keep the ENO property. And such a strategy indeed decreases the dissipation around the discontinuous region. In [1], Acker et al. present a improved fifth-order WENO-Z scheme through adding a new term to the fifth-order WENO-Z weights to further increase the relevance of less-smooth substencils. The improved scheme attains much better res-

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olution at the smooth parts of the solution, while keeping the same numerical stability of the original WENO-Z at shocks and discontinuities.

Based on the above status of research, it can be seen that improving the accuracy at critical points and increasing the contribution of the less smooth substencils are the two major methods to improve the performance of the conventional WENO-Z scheme. Above researchers mainly focus on two ways to improve third-order WENO-Z scheme achieving optimal order near critical points. One way is to modify the reference smoothness indicators τ of the WENO-Z scheme to achieve the optimal order near critical points like the WENO–NP3 scheme [17] and the WENO–NF3 scheme [5]. Another way is to modify the parameter ε to be $o(h^2)$ like the ESWENO scheme [19]. In the present work, we have devised an improved third-order WENO-Z scheme achieving optimal order near critical points through modifying the local smoothness indicators in the weighting function α_k and with the way of Taylor expansion of the local smoothness indicators for the third-order convergence.

2. Numerical methods

For the hyperbolic conservation law in the form

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0 \tag{1}$$

the flux function $f(u)$ can be split into two parts as $f(u) = f^+(u) + f^-(u)$ with $df^+(u)/du \geq 0$ and $df^-(u)/du \leq 0$. The semi-discrete form of Eq. (1) can be written as

$$\frac{du_i(t)}{\partial t} = -\frac{1}{h}\left(\bar{f}_{i+1/2} - \bar{f}_{i-1/2}\right) \tag{2}$$

where the numerical flux is $\bar{f}_{i+1/2} = \bar{f}^+_{i+1/2} + \bar{f}^-_{i+1/2}$. In this paper, only the positive part $\bar{f}^+_{i+1/2}$ $\bar{f}^+_{i+1/2}$ is described, and the superscript '+' is dropped for simplicity. The $\bar{f}^-_{i+1/2}$ is evaluated following the symmetric rule about $x_{i+1/2}$. h denotes the uniform grid size.

2.1. WENO-JS3 scheme

The second-order fluxes are defined on the two stencils $\{x_{i-1}, x_i\}$, $\{x_i, x_{i+1}\}$ as follows:

$$f_{0,i+1/2} = -\frac{1}{2}f_{i-1} + \frac{3}{2}f_i,\ f_{1,i+1/2} = \frac{1}{2}f_i + \frac{1}{2}f_{i+1} \tag{3}$$

The third-order WENO flux can then be written as

$$\bar{f}_{i+1/2} = \omega_0 f_{0,i+1/2} + \omega_1 f_{1,i+1/2} \tag{4}$$

where ω_0 and ω_1 are weight functions assigned to the two stencils, respectively. The classical weight functions proposed by Jiang and Shu in [8] are given by

$$\omega_k = \frac{\alpha_k}{\sum_{s=0}^{1}\alpha_s},\ k = 0, 1 \tag{5}$$

With

$$\alpha_k = \frac{d_k}{(\varepsilon+\beta_k)^2},\ d_0 = \frac{1}{3},\ d_1 = \frac{2}{3},\quad \beta_0 = (f_{i-1} - f_i)^2,\ \beta_1 = (f_i - f_{i+1})^2 \tag{6}$$

where the parameter ε in Eq. (6) is set to be 10^{-6} in the present work, as recommended in [8].

2.2. WENO–NP3 scheme

For the WENO–N3 scheme, similar to the classical weights, the weight functions are given by the Eq. (5), whereas, the function α_k proposed in [18] are different from their classical counterparts and defined as

$$\alpha_k = d_k\left(1 + \frac{\tau_N}{\beta_k + \varepsilon}\right),\ k = 0, 1 \tag{7}$$

$$\tau_N = \left|\frac{\beta_0 + \beta_1}{2} - \beta_3\right|,\ \varepsilon = 10^{-40} \tag{8}$$

Then, in order to achieve the designed convergence accuracy at the critical points, the WENO–NP3 scheme is proposed in [17]. For the WENO–NP3 scheme, the function α_k are defined as

$$\alpha_k = d_k\left(1 + \frac{\tau_{NP}}{\beta_k+\varepsilon}\right),\ k = 0, 1,\quad d_0 = \frac{1}{3},\ d_1 = \frac{2}{3} \tag{9}$$

$$\tau_{NP} = \left|\frac{\beta_0 + \beta_1}{2} - \beta_3\right|^p,\ \varepsilon = 10^{-40},\ p = 3/2 \tag{10}$$

where β_3 is the smoothness indicator of the whole stencil $\{x_{i-1}, x_i, x_{i+1}\}$ and which is expressed as [5]

$$\beta_3 = \frac{13}{12}(f_{i-1} - 2f_i + f_{i+1})^2 + \frac{1}{4}(f_{i-1} - f_{i+1})^2 \tag{11}$$

Taylor series expansion of β_3 at x_i can be written as

$$\begin{aligned}\beta_3 = {} & f_i'^2h^2 + \left(\frac{13}{12}f_i''^2 + \frac{1}{3}f_i'f_i'''\right)h^4 \\ & + \frac{1}{60}f_i'f_i^{(5)}h^6 + \frac{13}{72}f_i''f_i^{(4)}h^6 + \frac{1}{36}f_i'''^2h^6 \\ & + \frac{1}{2520}f_i'f_i^{(7)}h^8 + \frac{13}{2160}f_i''f_i^{(6)}h^8 + \frac{1}{360}f_i'''f_i^{(5)}h^8 \\ & + \frac{13}{1728}f_i^{(4)2}h^8 + o\left(h^{10}\right)\end{aligned} \tag{12}$$

2.3. The improved WENO scheme

2.3.1. Basic derivation

To achieve the desired convergence order at critical points for the WENO scheme, we introduced another improved WENO-Z scheme through modifying the local smoothness indicator β_k in Eq. (7) with the following form

$$\alpha_k = d_k\left(1 + \frac{\tau_N}{(\beta_k+\varepsilon)^p}\right),\ k = 0, 1,\quad d_0 = \frac{1}{3},\ d_1 = \frac{2}{3} \tag{13}$$

$$\tau_N = \left|\frac{\beta_0 + \beta_1}{2} - \beta_3\right|,\ \varepsilon = 10^{-40} \tag{14}$$

Notations. In the following section, we will verify that the parameter p should be $p <= 3/4$, in order to achieve the optimal order at the critical points.

Proof. The smoothness indicators of the candidate stencils in Eq. (6) can be expanded as a Taylor series at x_i