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$r-$adaptation for Shallow Water flows: conservation, well balancedness, efficiency

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ABSTRACT

We investigate the potential of the so-called "relocation" mesh adaptation in terms of resolution and efficiency for the simulation of free surface flows in the near shore region. Our work is developed in three main steps. First, we consider several Arbitrary Lagrangian Eulerian (ALE) formulations of the shallow water equations on moving grids, and provide discrete analogue in the Finite Volume and Residual Distribution framework. The compliance to all the physical constraints, often in competition, is taken into account. We consider different formulations allowing to combine volume conservation (DGCL) and equilibrium (Well-Balancedness), and we clarify the relations between the so-called pre-balanced form of the equations [7]), and the classical upwinding of the bathymetry term gradients [1]. Moreover, we propose a simple remap of the bathymetry based on high accurate quadrature on the moving mesh which, while preserving an accurate representation of the initial data, also allows to retain mass conservation within an arbitrary accuracy. Second, the coupling of the resulting schemes with a mesh partial differential equation is studied. Since the flow solver is based on genuinely explicit time stepping, we investigate the efficiency of three coupling strategies in terms of cost overhead w.r.t. the flow solver. We analyze the role of the solution remap necessary to evaluate the error monitor controlling the adaptation, and propose simplified formulations allowing a reduction in computational cost. The resulting ALE algorithm is compared with the *rezoning* Eulerian approach with interpolation proposed e.g. in Tang and Tang [17]. An alternative cost effective Eulerian approach, still allowing a full decoupling between adaptation and flow evolution steps is also proposed. Finally, a thorough numerical evaluation of the methods discussed is performed. Numerical results on propagation, and inundation problems shows that the best compromise between accuracy and CPU time is provided by a full ALE formulation. If a loose coupling with the mesh adaptation is sought, then the cheaper Eulerian approach proposed is shown to provide results quite close to the ALE.

1. Introduction

We investigate the potential benefits of r-adaptation techniques (or relocation adaptation) in the computation of propagation and interaction of free surface waves, including their runup on complex bathymetries. The main building blocks of our study are the following. First, we use the well known Shallow Water equations to model the hydrostatic free surface hydrodynamics in vicinity of the shore. The use of moving meshes will lead us to investigate various forms of the model equations in an Arbitrary-Lagrangian-Eulerian (ALE) setting. Second, the equations are coupled with a Laplacian-based adaptive mesh deformation technique. The coupling between flow evolution and mesh Partial Differential Equation (PDE) is discussed. Third, a thorough quantitative evaluation of the resulting algorithms on benchmarks involving wave propagation and inundation of complex bathymetries is performed.

The numerical approximation of Shallow Water flows is still a subject of intense research. For our purposes, the most interesting issue is the need of preserving, possibly to machine accuracy, the so-called lake at rest steady state. This property is known as C-property or *well balacedness* (WB). The initial work of [1] on the construction of well-balanced Finite-Volume approximations in one dimension, has been led throughout the years to many different results allowing the construction of unstructured mesh discretizations verifying the C-property via an appropriate coupling of the numerical flux and numerical source terms [2,3], or based on different forms of the equations, as the well-balanced form of [4–6], or the so-called *pre-balanced* form of Rogers et al. [7–9].

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These ideas have been also incorporated in Finite Element, Residual Distribution, and Discontinuous Galerkin methods (see e.g. [10–14] and references therein).

To enhance the resolution of complex wave patterns, and of the wetting/drying dynamics we study the use of mesh adaptation techniques based on nodes redistribution (or relocation). These are known as *r*-adaptation techniques. The reason for this choice is, on one hand the overhead represented by remeshing techniques [15,16] w.r.t. a single time step of a fully explicit discretization of the shallow water model, on the other the potential shown in the past for these techniques in e.g. [17], and in the numerous works of Budd and collaborators (see e.g. the review [18] and references therein). Nodal movement is obtained by solving an appropriate Moving Mesh Partial Differential Equation (MMPDE). Originally in [19] an equation for one dimensional grid movement was obtained from an integral statement of the equidistribution principle of De Boor [20]. Given the reference/computational coordinates $\boldsymbol{X}$ and the actual/physical one $\boldsymbol{x}$, the central idea is to find a transformation $\boldsymbol{x} = M(\boldsymbol{X}, t)$ that equidistributes some measure of the solution error (monitor function) on the reference domain. During the last decades theoretical arguments and experience lead to the design of quite general monitor functions which can ensure the adaptation to particular features of the solution; the arclenght-type monitor function of Winslow [21], based on solution gradients, is one of the most successful. Two dimensional MMPDE were later introduced in [22] based on a variational formulation of the equidistribution principle and on an analogy with the harmonic map of Dvinsky [23]. More recently, the use of Monge-Ampere equation for mesh movement have also attracted considerable interest, see [24]. In the present work we have used the two dimensional MMPDE of Ceniceros and Hou [25] which, while simple, prove to be computationally efficient and capable to follow complex flow evolutions up to small scale phenomena. This approach has been quite successful, and it has been used among the others, in [17,26,27].

The coupling of the flow solver with the mesh at each time step is non-trivial, as the mesh equations depend on the solution on the (unknown) adapted mesh. In particular the Shallow Water equations and the MMPDE can be either solved *simultaneously* or *alternately*. The latter has been successfully implemented by [22,28], showing a significant reduction of stiffness problems even if it can lead to a lag in the mesh movement with respect to the physical features. Depending on the framework in which we evolve the PDEs, two different *alternate* algorithms are tested at this point. If the PDEs are written in Eulerian framework one gets the *rezoning* method suggested in [17]. This approach, based on a sequence of mesh and flow iterations, uses the mesh solver as a black box, the flow equations being solved on a (different) fixed mesh at each time iteration. Its drawback is that, at each time iteration, mesh change is treated using a remap/interpolation of the flow variables from the old mesh to the new one. This may be quite expensive as it needs to guarantee the same properties as the flow solver itself (high order accuracy, non-oscillatory character/positivity preservation, C-property, mass conservation). At the opposite, once the grid has been adapted, one can evolve the flow in a reference framework which follows the transformation of the mesh using an Arbitrary-Lagrangian-Eulerian formulation of the flow equations. In this case, the mesh movement effects are incorporated in the discretization of the physical PDEs and the properties of the solution are only determined by the scheme.

However, a proper ALE form of the numerical discretization has to be used. In particular, a well known requirement for ALE discretizations is the compatibility with a Geometric Conservation Law (GCL), which guarantees that no artificial volume (*viz* mass) is produced in the computational domain due to mesh motion. The discrete counterpart of this property is known as the DGCL. (cf. the pioneering work [29,30] and the more recent [31] for an overview). For aeroelastic and aerodynamics simulations the ALE approach is very common for its simplicity to deal with the problem of moving boundaries, see [32]. In the moving mesh community, a non conservative form of the ALE coupling was extensively studied in [22] and [28] and it is referred to as the *quasi-Lagrange* approach. However, approximations of such a non conservative form of the ALE equations “hide” the DGCL and this in turns makes difficult to conserve the total domain volume along the simulation. More recently an *r*-adaptive ALE approach in conservation form, similar to the one used in this work, was employed in [33,34].

Ideally, in Shallow Water flows, we have to ensure the satisfaction of both a discrete analog of the GCL, and of the C-property, while still being able to conserve mass and momentum. A solution based on an ALE remap of the bathymetry has been suggested in [27]. However, unless such remap is very high order accurate, this quickly leads to a smoothing of the data, hence a re-initialization of the topography is required, spoiling mass conservation.

In this paper, we propose and evaluate simplified strategies allowing adaptive simulations of Shallow Water flows with wetting/drying fronts. In order to do this, we systematically review the forms of the equations which are best suited for the task of combining well-balancedness and DGCL on moving meshes; we use the resulting model equations to provide well-balanced high-order Finite Volume (FV) and Residual Distribution (RD) discretizations, clarifying the relations between the pre-balanced and well-balanced approaches; we provide a simple recipe to marry mass conservation and C-property on moving meshes using a re-interpolation of the nodal bathymetry based on accurate quadrature of the given bathymetric data; we define improved ad-hoc error estimators allowing to better track both smooth waves, and shorelines; finally, coupling strategies allowing cheaper and simpler interpolation algorithms in the adaptation phase, while retaining all the desired discrete properties, are evaluated in terms of CPU time for a given resolution, using standard benchmarks for near shore hydrodynamics.

The paper is organized as follows: Section 2 presents the general setting, and in particular it recalls the main forms and properties of the Shallow Water equations, and of a simpler scalar model used to simplify part of the discussion. The well-balanced numerical approximation of the PDEs in ALE form with Finite Volume and Residual distribution schemes is discussed in Section 3, with a discussion of the appropriate ALE form for balance laws. The moving mesh algorithm is presented in Section 4, with some details concerning the management of wet-dry areas in shallow water simulations. In Section 4.2 the interpolation strategy of [35] is presented as a conservative ALE remap. Three strategies to couple mesh movement and balance laws solution are presented in section Section 5: the rezoning algorithm of Tang coupled with the SW equations, see Zhou [35] (EUL1), an improved version of the above algorithm (EUL2) and the ALE coupling. Finally, Section 7 and 8 presents a thorough study of the coupled algorithms in terms of accuracy, and CPU time for both simple academic problems and for some classical benchmarks involving the long wave runup on complex bathymetries. The paper is ended by a summary of the main results, and by an overlook at future developments.

2. Problem setting and model equations

2.1. Shallow Water equations and lake at rest

Our final objective is the simulation of the propagation and runup of free surface waves in the near shore region. A good model for the physics of these waves is given by the nonlinear Shallow Water equations, providing a depth averaged description of the