Computers and Fluids 159 (2017) 81–94

Contents lists available at ScienceDirect

Computers and Fluids

journal homepage: www.elsevier.com/locate/compfluid

A new class of high-order weighted essentially non-oscillatory schemes for hyperbolic conservation laws

Fengxiang Zhao[a,b], Liang Pan[b], Zheng Li[b], Shuanghu Wang[b,*]

[a] *The Graduate School of China Academy of Engineering Physics, Beijing, China*
[b] *Institute of Applied Physics and Computational Mathematics, Beijing, China*

ARTICLE INFO

Article history:
Received 15 April 2017
Revised 17 August 2017
Accepted 18 September 2017
Available online 20 September 2017

Keywords:
WENO reconstruction
Hyperbolic conservation laws
Finite volume scheme

ABSTRACT

In this paper, a class of even order WENO schemes, including fourth-order, sixth-order and eighth-order schemes, are presented in finite volume framework for hyperbolic conservation laws. Instead of two upwind stencils in the classical WENO reconstruction [10] for each cell interface, a common symmetrical stencil is used in the reconstruction of the variables at its both sides. For each cell interface, the $2r$th order scheme shares the same number of cell averaged values with the $(2r-1)$th order classical WENO scheme. To suppress the spurious oscillation and improve the resolution in the region with discontinuities, a convex combination and a nonlinear weighting strategy are given with the unequal degree polynomials from the candidate sub-stencils. In smooth region, the current schemes achieve one order of improvement in accuracy compared with the classical WENO schemes. A variety of numerical tests are presented to validate the performance of the current schemes. Complex flow structures and discontinuities can be well resolved, and the robustness is as good as that of the classical WENO schemes.

1. Introduction

In past decades, there have been tremendous efforts on designing high order numerical methods for hyperbolic conservation laws and great success has been achieved. High order numerical methods were pioneered by Lax and Wendroff [12], and extended into the version of high resolution methods by van Leer [21], Harten [7] and other higher order versions, such as essentially non-oscillatory scheme (ENO) [8,18], weighted essentially non-oscillatory scheme (WENO) [10,13], Hermite weighted essentially non-oscillatory scheme (HWENO) [14,15], and discontinuous Galerkin scheme (DG) [4,5,16], etc.

The ENO and WENO schemes have been successfully applied for the compressible flows with strong shocks, contact discontinuities and complicated smooth structures. The ENO scheme was first introduced by Harten et al. [8] in the form of cell averaged variables. The key idea of ENO scheme is to use the “smoothest” stencil among several candidates to approximate the fluxes at cell interfaces to achieve high order accuracy and avoid spurious oscillations near discontinuities. Later, the flux version of ENO scheme [18] was introduced with TVD Runge–Kutta temporal discretization. However, the ENO scheme is not effective in terms of using only one stencil to approximate the fluxes at cell interfaces, and such an adaption of stencils is not necessary in smooth regions. To overcome these drawbacks while keeping the robustness and high order accuracy of ENO scheme, the WENO scheme was first introduced in [13]. Instead of approximating the pointwise values of the solution using only one of the candidate stencils, a convex combination of all the candidate stencils was used. Each candidate stencil is assigned a weight which determines the contribution of this stencil to the final approximation of pointwise values. The weights can be defined in such a way that it approaches certain optimal weights to achieve a higher order of accuracy in smooth regions, and the stencils which contain the discontinuities are assigned a nearly zero weight in the regions near discontinuities. A higher order of accuracy is obtained by emulating upstream scheme with the optimal weights away from the discontinuities, and the essentially non-oscillatory property is achieved near discontinuities. The flux of WENO scheme is smoother than that of the ENO scheme, and the smoothness enables us to prove convergence of WENO scheme for smooth solutions using Strang’s technique [10].

However, the optimal order of WENO scheme was not attained in [13], i.e. $(2r-1)$th-order with rth-order ENO scheme. A more detailed error analysis for WENO scheme was carried out, and a new WENO scheme (WENO-JS) including smoothness indicators and nonlinear weights was constructed in [10]. It was made possible to generalize the scheme to the fifth-order of accuracy. Later, very high order WENO schemes were developed as well [1]. The

* Corresponding author.
E-mail addresses: kobezhao@126.com (F. Zhao), panliangjlu@sina.com (L. Pan), zheng_li@iapcm.ac.cn (Z. Li), wang_shuanghu@iapcm.ac.cn (S. Wang).

https://doi.org/10.1016/j.compfluid.2017.09.013
0045-7930/

WENO-JS scheme may lose accuracy if the solution contains local smooth extrema. A new WENO scheme (WENO-M) was developed to overcome this problem by modifying the nonlinear weights by a mapping procedure. Meanwhile, the proposed mapping procedure is revealed to be computationally expensive. With a different weighting formulation, another version of the fifth-order WENO scheme (WENO-Z) was introduced in [2], in which a global higher order reference value was used for the smoothness indicator. As the improvement of WENO-Z scheme, WENO-Z+ scheme [3] was also developed. The WENO-M and WENO-Z and WENO-Z+ schemes turned out to be less dissipative than the classical WENO-JS scheme near smooth extrema. In the solution with high-frequency waves, they achieve noticeably higher amplitudes than WENO-JS scheme in a coarse grid. Some even-order schemes are developed for the effective direct numerical simulation of compressible turbulence in [24].

In this paper, a class of even order WENO schemes are presented in finite volume framework for hyperbolic conservation laws. Due to the widely usage of the classical fifth-order WENO scheme, the corresponding sixth-order scheme is introduced in detail and the fourth-order and eighth-order schemes are given briefly. In the classical WENO scheme [10], the values at both sides of each cell interface are reconstructed with two upwind stencils, while a common symmetrical stencil is used in the reconstruction of the variables at its both sides in the current approach. For each cell interface, the 2rth order scheme shares the same number of cell averaged values with the $(2r-1)$th order classical WENO scheme. To suppress the spurious oscillation and improve the resolution in the region with discontinuities, a convex combination and non-linear weighting strategy is given with the unequal degree polynomials from the candidate sub-stencils. For example, two quadratic and two cubic polynomials are used in the sixth-order scheme. Compared with the classical WENO reconstruction, higher resolution at discontinuities can be provided with one order of improvement for accuracy in the smooth region. A variety of numerical tests are presented to validate the performance of the current schemes. Complex flow structures and discontinuities can be well resolved, and the robustness is as good as that of classical WENO scheme.

This paper is organized as follows. In Section 2, the classical WENO scheme in finite volume framework is briefly reviewed. a class of even order WENO schemes are presented in Section 3 with some detailed analysis. Section 4 includes numerical examples to validate the current schemes. Section 5 is the conclusion.

2. Finite volume type WENO scheme

2.1. Finite volume method

In this paper, the following Euler equations are considered

$$\frac{\partial W}{\partial t}+\frac{\partial F(W)}{\partial x}=0, \tag{1}$$

where $W=(\rho,\rho U,\rho E)$, $F(W)=(\rho U,\rho U^2+p,(\rho E+p)U)$, $\rho E=\frac{1}{2}\rho U^2+\frac{p}{\gamma-1}$ and γ is the specific heat ratio. Integrating Eq. (1) over the computational cell $I_i=[x_{i-1/2},x_{i+1/2}]$, the semi-discretized form of finite volume scheme can be written as

$$\frac{\mathrm{d}W_i}{\mathrm{d}t}=\mathcal{L}_i(W)=-\frac{1}{\Delta x}[F(W(x_{i+1/2},t))-F(W(x_{i-1/2},t))], \tag{2}$$

where W_i is the cell averaged value on I_i, Δx is the cell size, and $F(W(x_{i+1/2},t))$ is the flux at cell interface $x=x_{i+1/2}$, which can be approximated by the numerical flux as follows

$$F(W(x_{i+1/2},t))\approx F_{i+1/2}=F(W^l_{i+1/2},W^r_{i+1/2}),$$

where $W^{l,r}_{i+1/2}$ are the reconstructed values at both sides of cell interface $x=x_{i+1/2}$. To fully discretize Eq. (1), the HLLC Riemann solver [20] is used for numerical fluxes, and the classical Runge–Kutta scheme [6] is used for temporal discretization. The spatial reconstruction is the main theme of this paper, which will be given in the following sections.

2.2. The classical WENO scheme

Before the new WENO scheme is introduced, we will briefly review the classical fifth-order WENO reconstruction [10,13]. Assume that W_i is the cell averaged value on I_i, and $W^l_{i+1/2}$ is the reconstructed values at the left side of cell interface $x=x_{i+1/2}$. The fifth-order WENO reconstruction for the value at the left side of cell interface $x=x_{i+1/2}$ is given as follows

$$W^l_{i+1/2}=\sum_{k=0}^{2}\delta_k w_{k,i+1/2},$$

where all quantities involved are taken as

$$w_{0,i+1/2}=\frac{1}{3}W_{i-2}-\frac{7}{6}W_{i-1}+\frac{11}{6}W_i,$$
$$w_{1,i+1/2}=-\frac{1}{6}W_{i-1}+\frac{5}{6}W_i+\frac{1}{3}W_{i+1},$$
$$w_{2,i+1/2}=\frac{1}{3}W_i+\frac{5}{6}W_{i+1}-\frac{1}{6}W_{i+2},$$

and $\delta_k,\ k=0,1,2$ is the nonlinear weight

$$d_0=\frac{1}{10},\ d_1=\frac{3}{5},\ d_2=\frac{3}{10}.$$

In order to deal with discontinuities, the local smoothness indicator $\beta_k,\ \ k=0,1,2$ is introduced [10]. For the fifth-order reconstruction, this definition yields

$$\beta_0=\frac{13}{12}(W_{i-2}-2W_{i-1}+W_i)^2+\frac{1}{4}(W_{i-2}-4W_{i-1}+3W_i)^2,$$
$$\beta_1=\frac{13}{12}(W_{i-1}-2W_i+W_{i+1})^2+\frac{1}{4}(W_{i-1}-W_{i+1})^2,$$
$$\beta_2=\frac{13}{12}(W_i-2W_{i+1}+W_{i+2})^2+\frac{1}{4}(3W_i-4W_{i+1}+W_{i+2})^2.$$

The most widely used is the WENO-JS nonlinear weight [10], which can be written as follows

$$\delta^{JS}_k=\frac{\alpha^{JS}_k}{\sum_{m=0}^{2}\alpha^{JS}_m},\ \ \alpha^{JS}_k=\frac{d_k}{(\beta_k+\epsilon)^2}.$$

In order to achieve better performance near smooth extrema, WENO-Z [2] reconstruction was developed, and the nonlinear weight for WENO-Z scheme is written as

$$\delta^{Z}_k=\frac{\alpha^{Z}_k}{\sum_{m=0}^{2}\alpha^{Z}_m},\ \ \alpha^{Z}_k=d_k\Big[1+\Big(\frac{\tau}{\epsilon+\beta_k}\Big)\Big],\ \ k=0,1,2,$$

where ϵ is a small positive number, τ makes the non-linear weights satisfying the sufficient condition to achieve the optimal order of accuracy, and $\tau=|\beta_0-\beta_2|$ is used for the fifth-order scheme. The procedure for $W^r_{i+1/2}$ at the right side of cell interface $x=x_{i+1/2}$ can be made up according to the symmetry property, and more detail can be found in [2,9,10].

3. New WENO schemes

In this section, a class of even order WENO schemes, including the fourth-order, sixth-order and eighth-order WENO schemes, will be proposed in the finite volume framework. For simplicity, the reconstruction procedure for the left side of cell interface $x_{i+1/2}$ is given in detail, and the procedure for the right side is given briefly for the sixth-order scheme.