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# Adaptive variational multiscale method for bingham flows

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ABSTRACT

The simulation of viscoplasitc flows is still attracting considerable attention in many industrial applications. However, the underlying numerical discretization and regularization may suffer from numerical oscillations, in particular for high Bingham and Reynolds numbers flows. In this work, we investigate the Variational Multiscale stabilized finite element method in solving such flows. We combined it with a posteriori error estimator for anisotropic mesh adaptation, enhancing the use of the Papanastasiou regularization. Computational results are compared to existing data from the literature and new results have demonstrated that the approach can be applied for Bingham numbers higher than 1000 yielding accurate predictions.



## 1. Introduction

Yield stress fluids represent materials that present viscoplastic behavior and have the property to flow when the material exceeds a yield limit. They are observed and studied in many research fields such as in geophysics to study avalanches [1], or magma and mud flows as in [2], blood rheology in [3], as well as in civil engineering to study for instance cement behavior [4]. Indeed, understanding yield stress mechanics improves the optimization of several industrial processes; however, accurate representation of such flow patterns remains a challenge for numerical methods.

Many kinds of yield stress fluids models exist in the literature. The most common is the Bingham model, which has the specificity to behave as a Newtonian fluid when the yield stress is exceeded:

$$\begin{cases} \tau = 2\left(\eta_p + \frac{\tau_0}{\bar{\dot{\varepsilon}}}\right)\dot{\boldsymbol{\varepsilon}} & \text{for } \bar{\tau} > \tau_0 \\ \dot{\boldsymbol{\varepsilon}} = 0 & \text{for } \bar{\tau} \leq \tau_0 \end{cases} \tag{1}$$

$\tau_0$ and $\eta_p$ represent yield stress and plastic viscosity. $\tau$ and $\dot{\boldsymbol{\varepsilon}}$ correspond respectively to extra-stress and strain rate tensors. $\bar{\dot{\varepsilon}}$ represent the second invariant of strain rate tensor, which is defined as:

$$\bar{\dot{\varepsilon}} = (2\dot{\varepsilon} : \dot{\varepsilon})^{\frac{1}{2}} \tag{2}$$

The choice of norm for $\tau$ is expressed as follow:

$$\bar{\tau} = \left(\frac{1}{2}\tau : \tau\right)^{\frac{1}{2}} \tag{3}$$

Numerical simulation offers a very flexible tool to model these kinds of fluid, and remains an inevitable step to study these complex fluids behavior. The remaining challenge is to construct efficient methods to capture such flow patterns in a robust and accurate way. In the literature, the computational domain may be discretized by different techniques in order to solve Bingham flows. In [5], a finite volume method is employed to discretize the equations which leads to approximate continuity and momentum equations on each control volumes. In [6], a Smoothed Particle Hydrodynamics (SPH) approach is applied, which can be viewed as a numerical scheme where the fluid flow is decomposed into discrete particles. The most common used method in the literature is the finite element formulation (see [7] and [8] for details). Nevertheless, the stability of the discrete formulation depends on appropriate compatibility restrictions on the choice of the finite element spaces [9]. The lack of stability manifests in uncontrollable oscillations that pollute the solution, in particular for high Bingham and Reynolds numbers.

On the other hand, we highlight another issue in the numerical simulation of a viscoplastic flow and is connected to the singularity of relations 1 and impossibility to determine stresses in the domains where the rate of deformation equals zero. In order to overcome these difficulties, various modifications, known as regularization methods have been introduced. We note two approaches, Bercovier-Engleman [10] and Papanastasiou [11], where the term

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$\eta_p + \tau_0/\dot{\bar{\varepsilon}}$ in relations 1 is replaced with $\eta_p + \tau_0/\sqrt{\dot{\bar{\varepsilon}}^2 + 1/m^2}$ (Bercovier-Engleman) or $\eta_p + \tau_0(1 - e^{-m\dot{\bar{\varepsilon}}})/\dot{\bar{\varepsilon}}$ (Papanastasiou) for an arbitrarily large regularizing parameter $m$. The performance and comparisons between these methods are summarized and analyzed in [12].

Despite the simplicity of implementing these models, some limitations still exist. The resolution is strongly dependent on the regularizing parameter $m$. Indeed, taking high values of this parameter encounters convergence issues whereas small values limit the flow prediction and the flow arrest is not controlled. One must find a compromise for choosing this parameter, in order to ensure reasonable computation time and good accuracy of the solution.

Several techniques have been developed aiming to increase this coefficient. One consists in applying a continuation method, which means to select dynamically $m$ and to keep it smaller during all the simulation. Another one found in the literature consists in performing a number of Picard iterations, and switch to the Newton method when a sufficiently good approximation to the solution is found [13]. One may also consider multiplier methods as alternatives for regularized models. It consists in computing the extra-stress tensor directly using minimization algorithms. The most useful way consists in employing Augmented Lagrangian method, coupled with Uzawa algorithm. These kinds of method reveals to be more accurate to determine flow arrests, but convergence may be slow, which can lead to unreasonable computational times.

In this work, we derive an adaptive Variational MultiScale (VMS) method for Bingham flows combined with a regularization method. The main reasons for this choice of adaptive variational approach are stability, robustness and computational efficiency. Indeed, mesh adaptation reveals to be a useful tool to improve accuracy, without reaching high computational times. It consists in refining the mesh in specific areas where physics reveals to be highly complex. In [14], an isotropic mesh adaptation is proposed and is based on the subdivision of a quadrilateral grid into subvolumes, each of them with the same mesh size. However, isotropic adaptation lacks in accuracy when the flow presents specific directional properties. We combine here the VMS formulation with an a posteriori error estimator for dynamic anisotropic mesh adaptation. It involves building a mesh based on a metric map. It provides both the size and the stretching of elements in a very condensed information data. Consequently, due to the presence of high gradients when using high values for the regularization coefficient, it provides highly stretched elements at the inner and the boundary layers, and thus yields an accurate modeling framework for Bingham flows as explained in [7]. The obtained system is then solved using a stabilized finite element method designed to handle the discontinuity on shear stress field. Indeed, it consists on the decomposition for both the velocity and the pressure fields into coarse/resolved scales and fine/unresolved scales, needed to deal with both high Bingham and Reynolds numbers.

We assess the behavior and accuracy of the proposed formulation in the simulation of three time-dependent challenging numerical examples, aiming for the first time to deal with high regularizing parameter (up to $10^6$), high Bingham (up to 2000) and Reynolds (up to 10000) numbers.

## 2. Governing equations

### 2.1. The incompressible Navier–Stokes equations

Let $\Omega \subset \mathbb{R}^n$ be the spatial domain at time $t \in [0, T]$, where $n$ is the space dimension. Let $\Gamma$ denote the boundary of $\Omega$. We consider the following velocity-pressure formulation of the Navier–Stokes equations governing unsteady incompressible flows:

$$\begin{cases} \rho(\delta_t \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v}) - \nabla\cdot\boldsymbol{\sigma} = \mathbf{f} & \text{in} \quad \Omega\times[0,T] \qquad (4)\\ \nabla\cdot\mathbf{v} = 0 & \text{in} \quad \Omega\times[0,T] \qquad (5)\end{cases}$$

where $\rho$ and $\mathbf{v}$ are the density and the velocity, $\mathbf{f}$ the body force vector per unity density and $\boldsymbol{\sigma}$ the stress tensor which reads:

$$\boldsymbol{\sigma} = 2\eta\ \boldsymbol{\varepsilon}(\mathbf{v}) - p\ \mathbf{I_d} \tag{6}$$

with $p$ and $\eta$ the pressure and the dynamic viscosity, $\mathbf{I_d}$ the identity tensor and $\boldsymbol{\varepsilon}$ the strain-rate tensor defined as

$$\boldsymbol{\varepsilon}(\mathbf{v}) = \frac{1}{2}(\nabla\mathbf{v} + {}^t\nabla\mathbf{v}) \tag{7}$$

Essential and natural boundary conditions for equation (4) are:

$$\mathbf{v} = \mathbf{g} \quad \text{on } \Gamma_g\times[0,T] \tag{8}$$

$$\mathbf{n}\cdot\boldsymbol{\sigma} = \mathbf{h} \quad \text{on } \Gamma_h\times[0,T] \tag{9}$$

$\Gamma_g$ and $\Gamma_h$ are complementary subsets of the domain boundary $\Gamma$. Functions $\mathbf{g}$ and $\mathbf{h}$ are given and $\mathbf{n}$ is the unit outward normal vector of $\Gamma$. As initial condition, a divergence-free velocity field $\mathbf{v}_0(\mathbf{x})$ is specified over the domain $\Omega_t$ at $t = 0$:

$$\mathbf{v}(\mathbf{x},0) = \mathbf{v}_0(\mathbf{x}) \tag{10}$$

### 2.2. Viscoplastic equation

Yield stress fluids present the particularity to have a yield limit $\tau_0$, which must be overcome in order the material starts to flow, otherwise, it has a perfectly rigid behavior. Bingham fluids are considered as ideal yield stress fluids because of their Newtonian behavior with yield limit overtaking. Constitutive equations of these kinds of fluid are shown here:

$$\begin{aligned} &\boldsymbol{\sigma} = \boldsymbol{\tau} - p\ \mathbf{I_d} \\ &\left(1 - \tfrac{\tau_0}{\overline{\tau}}\right)\boldsymbol{\tau} = 2\eta_p\boldsymbol{\varepsilon}(\mathbf{v}) \qquad \text{for } \overline{\tau} \geq \tau_0 \\ &\boldsymbol{\varepsilon}(\mathbf{v}) = 0 \qquad\qquad\qquad\ \ \text{for } \overline{\tau} < \tau_0 \end{aligned} \tag{11}$$

However, the extra stress tensor $\boldsymbol{\tau}$ is not explicitly defined under the yield stress value, and thus, constitutive equations are not continuous in all the fluid. By the way, the main challenge consists in taking into account the material behavior into motion and mass equations. In this paper, we use a regularization method, which consists in computing effective viscosity of the fluid. When the fluid is flowing, this viscosity must approach the plastic viscosity and when no deformations occur, it must be the maximum possible. Regularization methods aim to control and limit the maximum viscosity, in order to avoid convergence problems due to viscosity jumps.

Papanastasiou proposed a regularization which consists in expressing effective viscosity as an exponential function of shear rate [11]. Thus, we find the following expression for the effective viscosity $\eta_e$ of the fluid :

$$\eta_e = \eta_p + \frac{\tau_0}{\dot{\bar{\varepsilon}}}[1 - exp(-m\dot{\bar{\varepsilon}})] \tag{12}$$

$m$ corresponds to the Papanastasiou regularizing coefficient designed to control the yield limit: the greater $m$, the better we approach the classical Bingham model. However, as mentioned before, it may manifest in uncontrollable oscillations and non-convergence solution, in particular for high Bingham and Reynolds numbers.